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Abstract

Quality of decisions in inventory management problems depends on accuracy of parameter estimates used for decision making. In many situations, error in decision making is unavoidable. In such cases, understanding sensitivity of objective function to sub-optimal decisions is necessary for better implementation of the model. We study sensitivity of expected demand-supply mismatch cost to sub-optimal ordering decisions in the newsboy model. We demonstrate through establishing a lower bound for cost deviation that the newsboy model is sensitive to error in ordering decisions. We generalize our conclusions to the discrete case too.

1 Introduction

The newsboy problem is one of the most well-researched and widely applicable inventory management problems (Choi, 2012), much like the economic order quantity (EOQ) model. It is about finding the optimum order quantity of a product whose demand is unknown at the time of procurement decision and mismatch between demand and supply at the end of the selling season attracts penalty. This classic inventory problem was first addressed by Arrow et al. (1951) in their seminal paper "Optimal Inventory Policy".

The simplest case of this problem, which we refer to as the standard newsboy problem, can be found in textbooks on operations management and inventory management (e.g., Silver et al., 1998, chap. 10). Due to its wide applicability, different variants of the standard newsboy problem have been developed in past six decades. Good review of these works can be found in Silver et al. (1998), Khouja (1999), Qin et al. (2011), and Choi (2012).

The standard newsboy model and its extensions assume knowledge of certain parameters and the decision variable(s) depends on them, e.g., optimal order quantity in the standard newsboy problem depends on unit under-stocking cost, unit over-stocking cost, and demand distribution function. Same is true for other inventory optimization models. One common implementation issue with these models is the possibility of error in estimation of the model parameters. Sometimes, estimation error in one or more model parameters is unavoidable; then the chance of operational decision (derived using parameter estimates) being the optimal

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(calculated using the true values) is very little. Khanra & Soman (2013) discussed this issue for the newsboy model. In these situations, sensitivity analysis helps in understanding the impact of sub-optimal decisions on the objective function. Depending upon the severity of the situation, further actions can be taken to improve the decision making.

Though the newsboy problem is well researched, sensitivity analysis of the newsboy model has not received its due attention. Lapin (1988, chap. 6) mentioned the nature of change (positive or negative) of order quantity as cost parameters change. Order quantity is increasing in under-stocking cost and decreasing in over-stocking cost. Gerchak & Mossman (1992) identified the impact of demand variability on order quantity. It increases with demand variability if it is greater than the mean demand; the relation is opposite if it is lesser than the mean. Due to convex nature of the newsboy model (Muckstadt & Sapra, 2010, p. 123), cost always increases (or profit always decreases) whenever parameter estimation error occurs (irrespective of its nature). The work of Eeckhoudt et al. (1995) on risk-averse newsboy problem confirms the observations of Lapin (1988). Lau & Lau (2002) studied impact of demand variability on newsboy type retailers, their observations are similar to that of Gerchak & Mossman (1992).

All the studies mentioned in the last paragraph deal with the direction of change. Other components of sensitivity analysis, i.e., symmetry, magnitude, and distribution of deviation remained largely unexplored. Recent work by Khanra & Soman (2013) addresses the questions of symmetry and magnitude of cost deviation. They identified two main elements of sensitivity analysis of inventory optimization models like the newsboy model: i) sensitivity of objective function to sub-optimal decisions and ii) sensitivity of decision variable(s) to parameter estimation error. These elements can be studied separately as they are independent of each other. Focus of this study in on the former.

Khanra & Soman (2013) identified the link between symmetry (skewness) of cost deviation[1] and that of the demand density function. A symmetric demand density function leads to symmetric cost deviation; similar relations hold for left-skewness and right-skewness. As a consequence of this relation, cost deviation is right-skewed if $cf < 1/2$ and left-skewed if $cf > 1/2$ (cf is critical fractile[2]) for symmetric unimodal distributions (e.g., normal distribution). They demonstrated magnitude of cost deviation when demand is normally distributed and found the newsboy model to be sensitive to sub-optimal ordering decisions, much more sensitive than the EOQ model. They also explored order quantity deviation and identified mean demand as the most important parameter in influencing the order quantity.

The above mentioned findings regarding direction and symmetry of cost deviation of the newsboy model are valid in every situation. The same is not true for magnitude of cost deviation; the study of magnitude by Khanra & Soman (2013) is limited to normal demand distribution; though, they hinted that their findings may be valid for symmetric unimodal distributions. In

[1]Deviation of expected demand-supply mismatch cost from its minimum is referred to as cost deviation. Similarly, deviation of order quantity from its optimal is referred to as order quantity deviation.

[2]Optimal order quantity in the standard newsboy problem is given by $F(Q^*) = cf$, where F is the demand distribution function (Silver et al., 1998, chap. 10).

this work, we generalize their observations regarding magnitude of cost deviation by establishing a lower bound of cost deviation for unimodal demand distributions. We also generalize our conclusions to the discrete newsboy model by comparing it with its continuous counterpart.

The remainder of the paper is organized as follows. We begin with identifying some properties of unimodal distributions in Section 2. In Section 3, a lower bound of cost deviation is established and comparisons are drawn to the EOQ model. Section 4 deals with the discrete case. Finally, we conclude in Section 5.

Notations and assumptions

a	Lower limit of demand. $a \geq 0$.
b	Upper limit of demand. $a < b < \infty$.
r	Ratio of demand limits. $r = a/b$, $r \in [0, 1)$.
$F()$	Distribution function of stochastic demand. $F(a) = 0$, $F(b) = 1$. To ensure existence of density function, we assume absolute continuity of F (Royden, 2004, p. 110). We further assume strict monotony of F in $[a, b]$ so that Q^* is unique[3].
$f()$	Density function associated with F. Mere existence of f is assumed; hence, f can be discontinuous[4] at countable number of points in $[a, b]$.
μ	Mean demand.
c	Mode of unimodal demand. We assume that $a < c < b$.
m	Location indicator of the mode. $m = (c - a)/(b - a)$, $m \in (0, 1)$.
θ	Probability of demand not exceeding the mode of unimodal demand, i.e., $\theta = F(c)$. Since F is strictly increasing in $[a, b]$ and $a < c < b$, $\theta \in (0, 1)$.
c_u	Unit under-stocking cost. $c_u > 0$.
c_o	Unit over-stocking cost. $c_o > 0$.
cf	Critical fractile. $cf = c_u/(c_o + c_u)$, $cf \in (0, 1)$.
Q	Order quantity. $Q \geq 0$. Q^* denotes its optimal.
$C(Q)$	Demand-supply mismatch cost for a supply of Q.
δ_Q	Deviation of order quantity from its optimal.
$\delta_C(\delta_Q)$	Deviation of expected mismatch cost from its minimum for order quantity of δ_Q.

2 Properties of unimodal demand

We follow the definition of Gkedenko & Kolmogorov (1968, p. 157) for unimodal distribution, i.e., F is convex in $[a, c]$ and concave in $[c, b]$. This definition of unimodality is broader than

[3]Optimum order quantity in the newsboy model is given by $F(Q^*) = cf$ (Silver et al., 1998, chap. 10). A strictly increasing F, then, ensures uniqueness of Q^*. A strictly increasing F in $[a, b]$ also implies that $f(x) > 0$ for almost all $x \in (a, b)$ whenever it exists.

[4]This is a generalization of conventional assumption of continuity of f, a requirement for differentiability of $E[C(Q)]$ by Leibniz's rule (Protter & Morrey, 1977, p. 284).

the conventional definition involving the density function. For unimodal distributions, strict monotony of F need not be assumed separately[5]. The family of unimodal distributions with support $[a, b]$, mode c, and $F(c) = \theta$ is represented by $\mathcal{UD}_{a,b,c,\theta}$. Using the definition of unimodality and following equation, a partial bound for $F \in \mathcal{UD}_{a,b,c,\theta}$ can be established.

Given a, b, c, and θ, let us define F_0 as the equation of straight lines connecting $(a, 0)$, (c, θ) and (c, θ), $(b, 1)$, i.e.,

$$F_0(x) = \begin{cases} \frac{x-a}{c-a}\theta & \text{if } x \in [a, c) \\ 1 - \frac{b-x}{b-c}(1-\theta) & \text{if } x \in [c, b]. \end{cases} \quad (1)$$

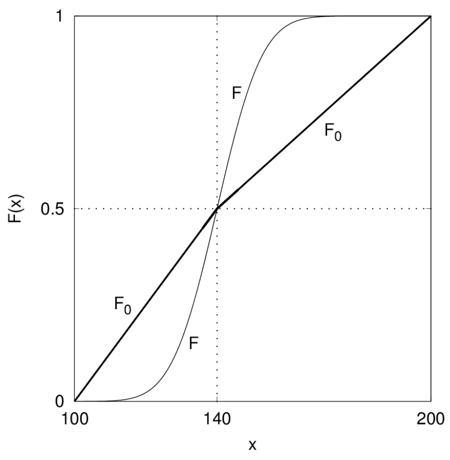
F_0 is a valid distribution; actually, $F_0 \in \mathcal{UD}_{a,b,c,\theta}$.

Lemma 1. *$F(x) \le F_0(x)$ if $x \in (a, c)$ and $F(x) \ge F_0(x)$ if $x \in [c, b)$ for every $F \in \mathcal{UD}_{a,b,c,\theta}$.*

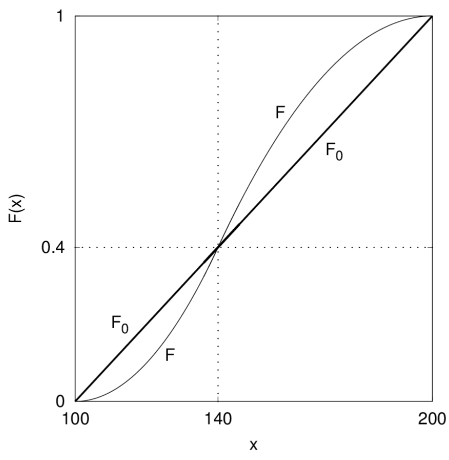
Above result has following consequence for the optimal order quantity in the newsboy model.

Corollary 1. *$Q^* \ge Q_0^*$ if $cf < \theta$ and $Q^* \le Q_0^*$ if $cf \ge \theta$ for every $F \in \mathcal{UD}_{a,b,c,\theta}$, where Q_0^* corresponds to $F_0 \in \mathcal{UD}_{a,b,c,\theta}$.*

See Appendix A for proofs of above results. Lemma 1 is depicted in Figure 1(a). The thick line corresponds to F_0. The thin curve corresponds to the truncated normal distribution. Note that a combination of any convex increasing curve connecting $(a, 0), (c, \theta)$ and any concave increasing curve connecting $(c, \theta), (b, 1)$ is a member of $\mathcal{UD}_{a,b,c,\theta}$.



(a) $\mathcal{UD}_{100,200,140,0.5}$



(b) $\mathcal{PUD}_{100,200,140}$

Figure 1: F_0 for unimodal and proportional unimodal distributions

Figure 1(b) represents a subclass of $\mathcal{UD}_{a,b,c,\theta}$, the family of proportional unimodal distributions (denoted by $\mathcal{PUD}_{a,b,c}$). The thin curve corresponds to the triangular distribution (a member of $\mathcal{PUD}_{a,b,c}$). Proportional unimodal distribution is defined as following.

[5]Unimodality ensures that $f(x) > 0 \; \forall x \in (a, b)$. If $f(a' > a) = 0$, $F(a') = 0$; then a can not be the lower limit. Similarly, if $f(b' < b) = 0$, $F(b') = 1$; then b can not be the upper limit. Positive f in (a, b) ensures strict monotony of F in $[a, b]$.

Definition 1. *A unimodal distribution F in $[a,b]$ with mode c is proportional if*

$$\frac{F(c+t_2)-F(c)}{F(c)-F(c-t_1)} = \frac{b-c}{c-a} \text{ whenever } \frac{t_2}{t_1} = \frac{b-c}{c-a} \ \forall t_1 \in (0, c-a], t_2 \in (0, b-c].$$

Putting $t_1 = c-a$ and $t_2 = b-c$ in the above definition, we get $\theta = F(c) = (c-a)/(b-a)$. Hence, a proportional unimodal distribution can be specified without θ. Let us represent the family of proportional unimodal distributions with support $[a,b]$ and mode c by $\mathcal{PUD}_{a,b,c}$. Note that every member of $\mathcal{UD}_{a,b,c,(c-a)/(b-a)}$ is not a proportional unimodal distribution; actually, $\mathcal{PUD}_{a,b,c} \subset \mathcal{UD}_{a,b,c,(c-a)/(b-a)}$. Triangular and uniform distributions fall in this subclass; in fact, $F_0 \in \mathcal{PUD}_{a,b,c}$ is the uniform distribution in $[a,b]$.

When $c = (a+b)/2$, $\mathcal{PUD}_{a,b,c}$ is the family of symmetric unimodal distributions in $[a,b]$. The normal distribution is a symmetric unimodal distribution. We introduce this subclass of unimodal distributions because a stronger lower bound of δ_C can be established for $F \in \mathcal{PUD}_{a,b,c}$.

3 Lower bound of cost deviation

We use the same ratio-based measure for deviation as Khanra & Soman (2013). $\delta_Q = (Q-Q^*)/Q^*$ and $\delta_C = (E[C(Q)] - E[C(Q^*)])/E[C(Q^*)]$. Using the standard results of the newsboy model, they derived the following expression for cost deviation.

$$\delta_C(\delta_Q) = \frac{\int_{Q^*}^{Q^*(1+\delta_Q)} \{F(x) - cf\} \mathrm{d}x}{(\mu - Q^*)cf + \int_a^{Q^*} F(x)\mathrm{d}x}. \tag{2}$$

The presence of F in the expression of $\delta_C(\delta_Q)$ makes the sensitivity analysis complicated. For this reason, Khanra & Soman (2013) have considered a specific case of F, the normal distribution. Here, we establish a lower bound for $\delta_C(\delta_Q)$ for any unimodal F.

Equation 2 can be rewritten as $\delta_C(\delta_Q) = N(\delta_Q)/D$, where

$$N(\delta_Q) = \int_{Q^*}^{Q^*(1+\delta_Q)} \{F(x) - cf\}\mathrm{d}x. \tag{3}$$

$$D = (\mu - Q^*)cf + \int_a^{Q^*} F(x)\mathrm{d}x. \tag{4}$$

Note that $N(\delta_Q)$ is non-negative[5] and D is positive[6]. We find lower bound of $\delta_C(\delta_Q)$ by finding an upper bound of D and a lower bound of $N(\delta_Q)$.

[6]Let $Q^*(1+\delta_Q) = Q$. If $\delta_Q < 0$, $Q < Q^*$ and $F(x) \leq cf$ $\forall x \in [Q, Q^*]$. Then $N(\delta_Q) = \int_Q^{Q^*} \{cf - F(x)\}\mathrm{d}x \geq 0$. Similarly, if $\delta_Q \geq 0$, $Q \geq Q^*$ and $F(x) \geq cf$ $\forall x \in [Q^*, Q]$. Then $N(\delta_Q) = \int_{Q^*}^{Q} \{F(x) - cf\}\mathrm{d}x \geq 0$.

[7]If $Q^* < \mu$, clearly, $D > 0$. If $Q^* \geq \mu$, $D = \int_a^{\mu} F(x)\mathrm{d}x + \int_{\mu}^{Q^*} \{F(x) - cf\}\mathrm{d}x > 0$.

3.1 Upper bound of the denominator

Proposition 1.

$$D < \begin{cases} D_0 + \frac{1}{2} cf(c-a)\theta & \text{if } cf < \theta \\ D_0 + \frac{1}{2}(1-cf)(b-c)(1-\theta) & \text{if } cf \geq \theta \end{cases}$$

$\forall F \in \mathcal{UD}_{a,b,c,\theta}$, where D_0 corresponds to $F_0 \in \mathcal{UD}_{a,b,c,\theta}$.

See Appendix B for a proof of the above result. Now, we find expressions for Q_0^* and D_0 for later use. Putting $F_0(Q_0^*) = cf$ in (1), we get

$$Q_0^* = \begin{cases} a + \frac{cf}{\theta}(c-a) & \text{if } cf < \theta \\ b - \frac{1-cf}{1-\theta}(b-c) & \text{if } cf \geq \theta \quad \text{for } F_0 \in \mathcal{UD}_{a,b,c,\theta}. \end{cases} \tag{5}$$

Using (1), (4), and (5),

$$D_0 = \begin{cases} \frac{cf}{2}\left[\frac{(1-cf)-(1-\theta)^2}{\theta}(c-a) + (1-\theta)(b-c)\right] & \text{if } cf < \theta \\ \frac{1-cf}{2}\left[\theta(c-a) + \frac{cf-\theta^2}{1-\theta}(b-c)\right] & \text{if } cf \geq \theta \quad \text{for } F_0 \in \mathcal{UD}_{a,b,c,\theta}. \end{cases} \tag{6}$$

Details of the above expressions can be found in Appendix C.

Putting D_0 of (6) into Proposition 1,

$$D < \begin{cases} \frac{cf}{2}\left[\left(2 - \frac{cf}{\theta}\right)(c-a) + (1-\theta)(b-c)\right] & \text{if } cf < \theta \\ \frac{1-cf}{2}\left[\theta(c-a) + \left(2 - \frac{1-cf}{1-\theta}\right)(b-c)\right] & \text{if } cf \geq \theta \quad \forall\ F \in \mathcal{UD}_{a,b,c,\theta}. \end{cases} \tag{7}$$

Using the defining property of proportional unimodal distribution, a stronger upper bound of D can be established for $F \in \mathcal{PUD}_{a,b,c}$.

Proposition 2.

$$D \leq \begin{cases} D_0 - \frac{1}{2} cf(a+b-2c)^- & \text{if } cf < \frac{c-a}{b-a} \\ D_0 + \frac{1}{2}(1-cf)(a+b-2c)^+ & \text{if } cf \geq \frac{c-a}{b-a} \end{cases}$$

$\forall F \in \mathcal{PUD}_{a,b,c}$, where D_0 corresponds to $F_0 \in \mathcal{PUD}_{a,b,c}$.

See Appendix D for a proof of the above result. For symmetric unimodal demand (the special case of proportional unimodal demand when $a + b = 2c$), $D \leq D_0$. Superiority of upper bound of D in Proposition 2 over that in Proposition 1 can be easily verified (see Appendix E).

Now, we find Q_0^* and D_0 for $F \in \mathcal{PUD}_{a,b,c}$. Putting $\theta = (c-a)/(b-a)$ in (5) and (6),

$$Q_0^* = a + cf(b-a) \quad \text{for } F_0 \in \mathcal{PUD}_{a,b,c}. \tag{8}$$

$$D_0 = \frac{1}{2}cf(1-cf)(b-a) \quad \text{for } F_0 \in \mathcal{PUD}_{a,b,c}. \tag{9}$$

Putting D_0 of (9) into Proposition 2,

$$D \leq \begin{cases} \frac{cf}{2}\left[(1-cf)(b-a) - (a+b-2c)^-\right] & \text{if } cf < \frac{c-a}{b-a} \\ \frac{1-cf}{2}\left[cf(b-a) + (a+b-2c)^+\right] & \text{if } cf \geq \frac{c-a}{b-a} \quad \forall\ F \in \mathcal{PUD}_{a,b,c}. \end{cases} \tag{10}$$

3.2 Lower bound of the numerator

We find lower bound of $N(\delta_Q)$ using properties associated with unimodality of F. Properties of the integrand in (3), $F(x) - cf$ changes with the location of x w.r.t. a, b, c, thereby changing the lower bound of $N(\delta_Q)$. Theoretically, Q can assume any non-negative value. However, if a_o and b_o are observed lowest and highest demands, it is very unlikely that one will order a quantity that is outside the observed demand range[8]. Since $[a_o, b_o] \subseteq [a, b]$, we can safely assume that $Q \in [a, b]$. With this assumption, following lower bound of $N(\delta_Q)$ can be established (see Appendix F for a proof).

Proposition 3. *For every $F \in \mathcal{UD}_{a,b,c,\theta}$, provided $Q = Q^*(1+\delta_Q) \in [a, b]$,*

$$N(\delta_Q) \geq \frac{Q_0^* cf}{2}\left[\frac{c}{c-a}\delta^2 + \frac{1-\theta}{cf}\left\{\frac{Q_0^*}{b-c}(\delta_Q^2 - \delta^2) + 2\left(\frac{\theta - cf}{1-\theta} - \frac{c - Q_0^*}{b-c}\right)(\delta_Q - \delta)\right\}\right]$$

if $cf < \theta$, where $\delta = \min\{\delta_Q, c/Q_0^* - 1\}$.

$$N(\delta_Q) \geq \frac{c(1-cf)}{2}\left[\frac{c}{b-c}\delta^2 + \frac{\theta}{1-cf}\left\{\frac{Q_0^*}{c-a}(\delta_Q^2 - \delta^2) + 2\left(\frac{cf-\theta}{\theta} - \frac{Q_0^* - c}{c-a}\right)(\delta - \delta_Q)\right\}\right]$$

if $cf \geq \theta$, where $\delta = \max\{\delta_Q, c/Q_0^* - 1\}$.

3.3 The lower bound

If D_{ub} is an upper bound of D, by Proposition 3,

Proposition 4. *For every $F \in \mathcal{UD}_{a,b,c,\theta}$, provided $Q = Q^*(1+\delta_Q) \in [a, b]$,*

$$\delta_C(\delta_Q) \geq \frac{Q_0^* cf}{2D_{ub}}\left[\frac{c}{c-a}\delta^2 + \frac{1-\theta}{cf}\left\{\frac{Q_0^*}{b-c}(\delta_Q^2 - \delta^2) + 2\left(\frac{\theta - cf}{1-\theta} - \frac{c - Q_0^*}{b-c}\right)(\delta_Q - \delta)\right\}\right]$$

if $cf < \theta$, where $\delta = \min\{\delta_Q, c/Q_0^* - 1\}$.

$$\delta_C(\delta_Q) \geq \frac{c(1-cf)}{2D_{ub}}\left[\frac{c}{b-c}\delta^2 + \frac{\theta}{1-cf}\left\{\frac{Q_0^*}{c-a}(\delta_Q^2 - \delta^2) + 2\left(\frac{cf-\theta}{\theta} - \frac{Q_0^* - c}{c-a}\right)(\delta - \delta_Q)\right\}\right]$$

if $cf \geq \theta$, where $\delta = \max\{\delta_Q, c/Q_0^* - 1\}$.

(7) and (10) provide D_{ub} for $F \in \mathcal{UD}_{a,b,c,\theta}$ and $F \in \mathcal{PUD}_{a,b,c}$ respectively. We can express

[8] This may not be true if number of observations is very few.

coefficients in Proposition 4 using the ratios, $r = a/b$, $m = (c-a)/(b-a)$, $cf = c_u/(c_o + c_u)$, and θ as following.

For every $F \in \mathcal{UD}_{a,b,c,\theta}$, provided $Q = Q^*(1+\delta_Q) \in [a, b]$,

$$\delta_C(\delta_Q) > k_0[k_1\delta^2 + k\{k_2(\delta_Q^2 - \delta^2) + 2k_3(\delta_Q - \delta)\}]. \tag{11}$$

If $cf < \theta$, $k = (1-\theta)/cf$ and $\delta = \min\{\delta_Q, q\}$, whereas $k = \theta/(1-cf)$ and $\delta = \max\{\delta_Q, q\}$ if $cf \geq \theta$. The constants, k_0, k_1, k_2, k_3, and q can be expressed as

Constant	when $cf < \theta$	when $cf \geq \theta$
$t,\ k_0$	$\dfrac{cf}{\theta},\ \dfrac{r/(1-r)+tm}{(2-t)m+(1-\theta)(1-m)}$	$\dfrac{1-cf}{1-\theta},\ \dfrac{r/(1-r)+m}{(2-t)(1-m)+\theta m}$
$k_1,\ k_2$	$\dfrac{r+m(1-r)}{m(1-r)},\ \dfrac{r+tm(1-r)}{(1-m)(1-r)}$	$\dfrac{r+m(1-r)}{(1-m)(1-r)},\ \dfrac{1-t(1-m)(1-r)}{m(1-r)}$
$k_3,\ q$	$\dfrac{(\theta-cf)(\theta-m)}{\theta(1-\theta)(1-m)},\ \dfrac{m(1-t)}{r/(1-r)+tm}$	$\dfrac{(cf-\theta)(m-\theta)}{\theta(1-\theta)m},\ \dfrac{-(1-m)(1-t)}{1/(1-r)-t(1-m)}.$

For proportional unimodal distribution, $\theta = m$. Then k_3 in (11) vanishes and $k \times k_2$ is the new k_2. For every $F \in \mathcal{PUD}_{a,b,c}$, provided $Q = Q^*(1+\delta_Q) \in [a, b]$,

$$\delta_C(\delta_Q) \geq k_0\{k_1\delta^2 + k_2(\delta_Q^2 - \delta^2)\}. \tag{12}$$

If $cf < \theta$, $\delta = \min\{\delta_Q, q\}$, whereas $\delta = \max\{\delta_Q, q\}$ if $cf \geq \theta$. The constants, k_0, k_1, k_2, and q can be expressed as

Constant	when $cf < m$	when $cf \geq m$
$k_0,\ k_1$	$\dfrac{r/(1-r)+cf}{(1-cf)-(1-2m)^-},\ \dfrac{r+m(1-r)}{m(1-r)}$	$\dfrac{r/(1-r)+m}{cf+(1-2m)^+},\ \dfrac{r+m(1-r)}{(1-m)(1-r)}$
$k_2,\ q$	$\dfrac{r+cf(1-r)}{cf(1-r)},\ \dfrac{(m-cf)(1-r)}{r+cf(1-r)}$	$\dfrac{r+cf(1-r)}{(1-cf)(1-r)},\ -\dfrac{(cf-m)(1-r)}{r+cf(1-r)}.$

The lower bound of $\delta_C(\delta_Q)$ depends on four factors (cf, θ, m, r) that takes values between 0 and 1. This enables us to construct different scenarios by changing these factors and study behaviour of the lower bound. For proportional unimodal distributions, θ is not required for scenario construction. Even for general unimodal distributions, we do not need to study scenarios with high difference between θ and m; those scenarios are rather impractical.

Figure 2 show the lower bound of $\delta_C(\delta_Q)$ in different scenarios. Each diagram corresponds to a combination of r and cf values; three values, 0.25, 0.5, and 0.75 (low, medium, and high) have been considered for both. $m = 0.5$ is taken.

In each diagram, solid curves correspond to the lower bounds for different θ values for $F \in \mathcal{UD}_{a,b,c,\theta}$; $\theta = m - 0.1$ is indicated by red, $\theta = m$ is indicated by blue, and $\theta = m + 0.1$ is

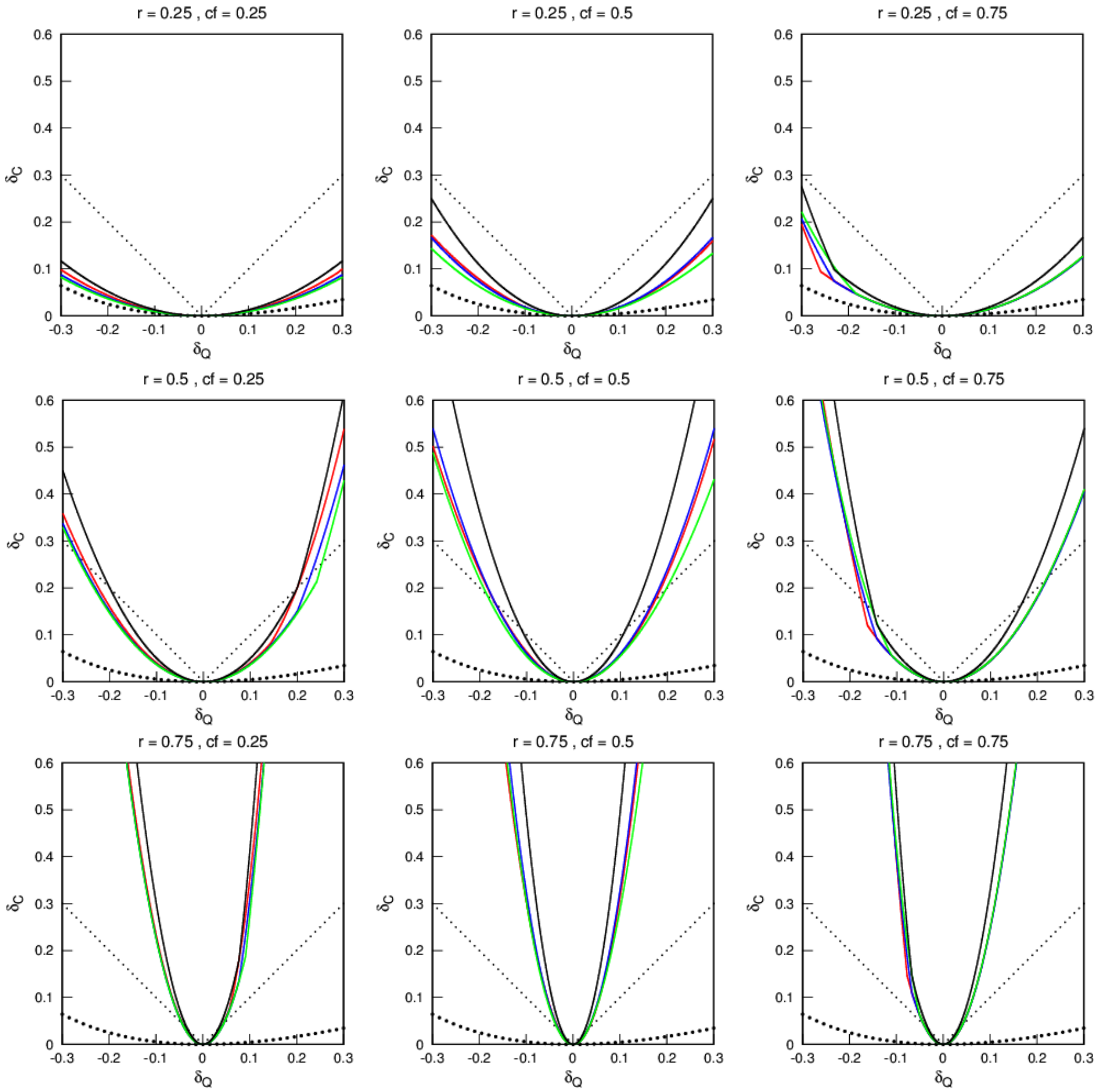


Figure 2: Lower bounds of $\delta_C(\delta_Q)$ when $m = 0.5$

indicated by green. The solid curve in black is the lower bound for $F \in \mathcal{PUD}_{a,b,c}$.

We also indicate $\delta_C(\delta_Q)$ for the EOQ model (thick dotted curve) for easy comparison; $\delta_C(\delta_Q) = \delta_Q^2/\{2(1+\delta_Q)\}$ for the EOQ model (Nahmias, 2001, p. 208). Dotted lines with slope 1 and -1 separate the error "dampening" and "amplifying" zones. If a curve (or part of it) lies below these lines, the magnitude of the output error is less than that of the input error (dampening of error). Conversely, if the curve (or part of it) lies above these lines, the output error is more in magnitude than the input error (amplification of error).

Key observations

i) Lower bound curves are steeper than the EOQ curve. Most part of the lower bound curves lie above the ± 1 slope lines for high r.

ii) Lower bound curves become steeper with increase in r. Other factors do not show strong influence on the lower bound.

Our observations are similar to that of Khanra & Soman (2013). Greater steepness of the lower bound curves compared to the EOQ curve conclusively demonstrates that the newsboy model is more sensitive to error in order quantity than the EOQ model. Like the EOQ curve, ± 1 slope lines act as benchmark. Locations of the lower bound curves imply that amplification of error occurs in many situations.

From the behaviour of the lower bound curves as r changes, we can conclude that robustness of the newsboy model decreases with increase in r. This behaviour can be explained by flattening of the density function due to decreased r (as demand range increases). Then the numerator of cost deviation, $N(\delta_Q) = \int_{Q^*}^{Q} \{F(x) - cf\} \mathrm{d}x = \int_{Q^*}^{Q} (Q - x) f(x) \mathrm{d}x$ decreases, thereby decreasing $\delta_C(\delta_Q)$. The effect reverses when r increases.

The special case of $r = 0$

Unlike m, θ, cf, which are unlikely to take extreme values, r can assume very low value. Figure 3 demonstrates the special case of $r = 0$. Construction of the diagrams is very similar to that of Figure 2. As expected, the lower bound of cost deviation decreases, but still remains at par with cost deviation of the EOQ model.

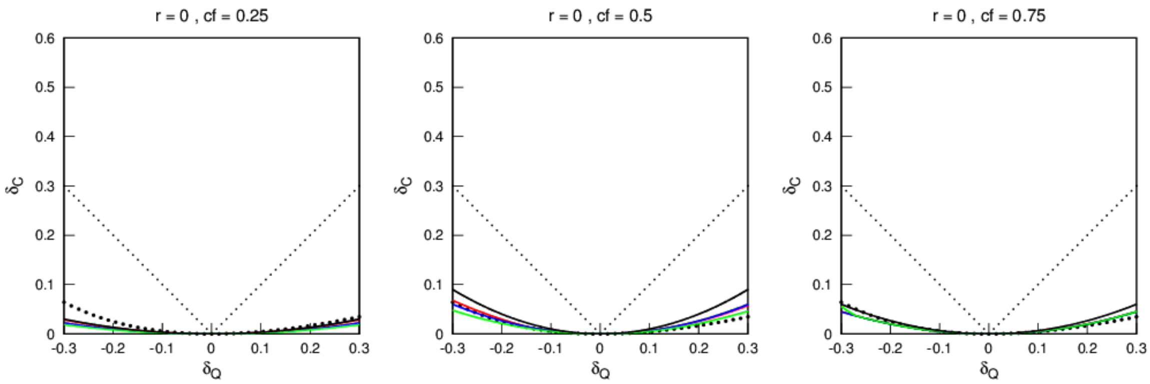


Figure 3: Lower bounds of $\delta_C(\delta_Q)$ when $r = 0$ and $m = 0.5$

In our demonstration, we do not show the lower bound for low and high values of m. Additional figures at the end of this document exhibit the lower bound for $m = 0.35$ and $m = 0.65$. Construction of those diagrams are same as Figure 2. They exhibit similar behaviour of the lower bound as observed in Figure 2.

4 The discrete case

In the previous section, we have generalized the conclusion of Khanra & Soman (2013) that the newsboy model is more sensitive than the EOQ model to unimodal demand distributions. In this section, we generalize this further to include the discrete case.

In the discrete version of the newsboy model, both stochastic demand and order quantity are integer valued. Expected mismatch cost for order quantity, Q is given by

$$E[C(Q)] = \sum_{i=a}^{Q-1} c_o(Q-i)p(i) + \sum_{i=Q+1}^{b} c_u(i-Q)p(i) \tag{13}$$

By marginal analysis (Muckstadt & Sapra, 2010, chap. 5), Q^* satisfies

$$P(Q^*-1) < cf \text{ and } P(Q^*) \geq cf. \tag{14}$$

The expression for cost deviation is not simple. Sensitivity analysis results of the continuous case can not be established for the discrete case in the same manner. Here, we construct a continuous equivalence of a given discrete newsboy model and demonstrate that cost deviation of the discrete problem can be approximated by that of the equivalent continuous problem. Then results applicable for the continuous equivalence can be extended to the discrete case.

4.1 Continuous equivalence

Continuous equivalence of a discrete newsboy model is a continuous newsboy model, where cost parameters remain same, order quantity is allowed to assume non-integer values, and the discrete demand is converted into a continuous demand. There are multiple ways of constructing the continuous demand; we use the following one.

Definition 2. *Continuous equivalence of a discrete demand with support a, b and mass function p is a continuous demand with support $[a-1/2, b+1/2]$ and density function f defined as: $f(x) = p(i)$ if $x \in (i-1/2, i+1/2]$ for $i = a, a+1, \ldots, b$ and $f(a-1/2) = p(a)$.*

Above definition yields a valid density in $[a-1/2, b+1/2]$ as the corresponding distribution function (it is defined as f is continuous almost everywhere) is increasing, continuous, and

$$F(a-1/2) = 0,\ F(b+1/2) = \int_{a-1/2}^{b+1/2} f(x)\mathrm{d}x = \sum_{i=a}^{b} \int_{i-1/2}^{i+1/2} p(i)\mathrm{d}x = \sum_{i=a}^{b} p(i) = 1.$$

In fact, F is strictly increasing. All of our assumptions about the distribution and density functions for the continuous case are satisfied by this continuous equivalence. Hence, this continuous equivalence, if unimodal, admits the lower bound of Proposition 4. It can be shown that continuous equivalence of a discrete unimodal demand is unimodal (see Appendix G).

Let us denote that mismatch cost and optimal order quantity of the continuous equivalence by C_{eq} and Q^*_{eq}. Following results establish the links between i) Q^* and Q^*_{eq}, ii) $E[C(Q^*)]$ and $E[C_{eq}(Q^*_{eq})]$, and iii) $E[C(Q)]$ and $E[C_{eq}(Q)]$ for integer Q. These results allows us to approximate $\delta_C(\delta_Q)$ by its continuous equivalence, $\delta_{C_{eq}}(\delta_Q)$.

Before identifying above mentioned relations, let us identify the link between F and P. For any integer $Q \in \{a, a+1, \dots, b\}$,

$$F(Q+1/2) = \int_{a-1/2}^{Q+1/2} f(x)\mathrm{d}x = \sum_{i=a}^{Q} \int_{i-1/2}^{i+1/2} p(i)\mathrm{d}x = \sum_{i=a}^{Q} p(i) = P(Q). \tag{15}$$

Lemma 2. $Q^*_{eq} \in (Q^* - 1/2, Q^* + 1/2]$.

The above result is easy to verify. If Q^* is the optimal order quantity for the discrete case, by (14) and (15), $F(Q^* + 1/2) = P(Q^*) \geq cf$ and $F(Q^* - 1/2) = P(Q^* - 1) < cf$. Since F is continuous, the intermediate value theorem (Protter & Morrey, 1977, p. 61) tells that $\exists\, Q \in (Q^* - 1/2, Q^* + 1/2]$ such that $F(Q) = cf$. Hence, $Q^*_{eq} \in (Q^* - 1/2, Q^* + 1/2]$.

We can find following expression for Q^*_{eq} using Lemma 2.

$$cf = F(Q^*_{eq}) = F(Q^* - 1/2) + \int_{Q^*-1/2}^{Q^*_{eq}} f(x)\mathrm{d}x = P(Q^* - 1) + p(Q^*)\{Q^*_{eq} - (Q^* - 1/2)\}.$$

$$\Rightarrow Q^*_{eq} = (Q^* - 1/2) + \frac{cf - P(Q^* - 1)}{p(Q^*)}. \tag{16}$$

Lemma 3. *For any $Q_0 \in \{a, a+1, \dots, b\}$ and $Q \in (Q_0 - 1/2, Q_0 + 1/2]$,*

$$\frac{E[C_{eq}(Q)] - E[C(Q_0)]}{c_o + c_u} = \frac{1}{2}(1-d)^2 p(Q_0) + (d - 1/2)\{P(Q_0) - cf\}, \text{ where } d = Q - (Q_0 - 1/2).$$

A proof of the above lemma appears in Appendix H. The above result is not much meaningful in itself; however, it has some interesting consequences. These corollaries help in developing the approximation for cost deviation.

Corollary 2. $E[C_{eq}(Q)] = E[C(Q)] + (c_o + c_u)p(Q)/8$ *for* $Q \in \{a, a+1, \dots, b\}$.

An integer Q in Lemma 3 means $Q_0 = Q$ and $d = 1/2$, which gives us the above result. See Appendix H for a proof of the next corollary.

Corollary 3. $0 < E[C_{eq}(Q^*_{eq})] - E[C(Q^*)] \leq (c_o + c_u)p(Q^*)/8$.

Let $z \in \mathbb{Z} \setminus \{0\}$ and $z \geq -Q^*$. Then a sub-optimal order quantity can be written as $Q = Q^* + z$. Deviation in order quantity is of the form: $\delta_Q = z/Q^*$. By Corollary 2 and 3,

$$\delta_C(\delta_Q) = \frac{E[C(Q)]}{E[C(Q^*)]} - 1 = \frac{E[C_{eq}(Q)] - (c_o + c_u)p(Q)/8}{E[C_{eq}(Q^*_{eq})] - \alpha(c_o + c_u)p(Q^*)/8} - 1, \text{ where } \alpha \in [0, 1].$$

$(c_o + c_u)p(Q)/8$ and $\alpha(c_o + c_u)p(Q^*)/8$ are positive and are likely to be much smaller than $E[C_{eq}(Q)]$ and $E[C_{eq}(Q^*_{eq})]$. Furthermore, $(c_o + c_u)p(Q)/8 \geq \alpha(c_o + c_u)p(Q^*)/8$ like $E[C_{eq}(Q)] \geq E[C_{eq}(Q^*_{eq})]$. Then

$$\delta_C(\delta_Q) \approx \frac{E[C_{eq}(Q)]}{E[C_{eq}(Q^*_{eq})]} - 1 = \delta_{C_{eq}}\left(\frac{Q}{Q^*_{eq}} - 1\right) = \delta_{C_{eq}}\left(\frac{Q^*}{Q^*_{eq}}(1+\delta_Q) - 1\right).$$

By Lemma 2, $Q^*/Q^*_{eq} \in [1 - 0.5/Q^*_{eq}, 1 + 0.5/Q^*_{eq})$. If $Q^*_{eq} = 10$, $Q^*/Q^*_{eq} \in [0.95, 1.05)$ and if $Q^*_{eq} = 50$, $Q^*/Q^*_{eq} \in [0.99, 1.01)$. We take $Q^*/Q^*_{eq} \approx 1$. Then $\delta_C(\delta_Q) \approx \delta_{C_{eq}}(\delta_Q)$.

Next, we test performance of the above approximation.

4.2 Sample calculation: Poisson distribution

We test the approximation for Poisson distribution. It is one of the most popular distributions for modelling discrete demand (Silver et al., 1998, p. 122). To find $\delta_C(\delta_Q)$ and $\delta_{C_{eq}}(\delta_Q)$ for given δ_Q, we need to know $E[C(Q^*)]$, $E[C(Q)]$, $E[C_{eq}(Q^*_{eq})]$, and $E[C_{eq}(Q_{eq})]$, where $Q = Q^*(1+\delta_Q)$ and $Q_{eq} = Q^*_{eq}(1+\delta_Q)$. Like the continuous case, it is reasonable to assume that $a \leq Q \leq b$. Then we can expect $Q_{eq} = Q^*_{eq}(1+\delta_Q) \in [a - 1/2, b + 1/2]$. We get Q^* and Q^*_{eq} by (14) and (16) respectively. Then we can find $E[C(Q^*)]$ and $E[C(Q)]$ using (13). We can find $E[C_{eq}(Q^*_{eq})]$ and $E[C_{eq}(Q_{eq})]$ using Lemma 3 and (13).

$\delta_C(\delta_Q)$ and $\delta_{C_{eq}}(\delta_Q)$ do not depend on c_o, c_u "directly", they depend on cf. We test performance of the approximation for three values of cf $(0.25, 0.5, 0.75)$. Since discrete modelling is used for slow moving items (lower demand), we consider small values of μ $(25, 50, 100)$. For Poisson distribution, $a = 0$ and we take $b = 500$ (distribution is right-truncated). Figure 4 shows the $\delta_C(\delta_Q)$ plots; dots represent $\delta_C(\delta_Q)$ for $\delta_Q = z/Q^*$ and solid curves represent $\delta_{C_{eq}}(\delta_Q)$. A flatter curve and nearby dots correspond to lower mean.

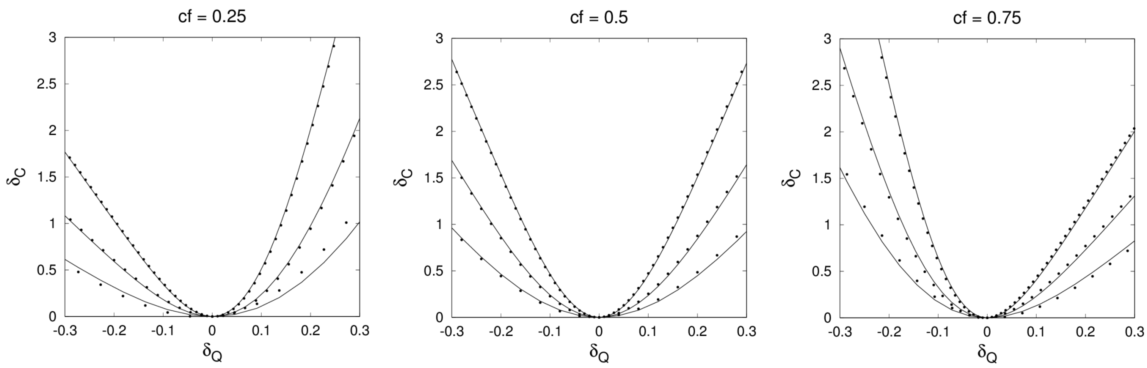


Figure 4: $\delta_C(\delta_Q)$ for Poisson distribution

It can be observed that $\delta_C(\delta_Q)$ plots for the discrete cases and their continuous equivalences are very close to each other in every situation. In some cases, they are visibly inseparable. Little mismatch is observed for $\mu = 25$ when $cf = 0.25$. In general, the approximation works well.

Performance of the approximation gets better as demand increases.

Due to similarity between the cost deviations for a discrete case and its continuous equivalence, we can conclude for the discrete case by studying its continuous equivalence. It has already been established that continuous equivalence of a discrete unimodal distribution is unimodal. Hence, the lower bound of cost deviation (and associated conclusions) in Proposition 4 is applicable for the discrete case with unimodal demand (with minor adjustments though).

5 Conclusion

Literature on sensitivity analysis of the newsboy model is not comprehensive; our understanding is limited to direction of cost and order quantity deviations, symmetry (skewness) of cost deviation, and magnitude of cost deviation for normal demand distribution. Our work contributes to the study of magnitude of cost deviation by generalizing it beyond normal distribution.

We establish a lower bound of cost deviation for the family of unimodal distributions (Proposition 4). A stronger lower bound is established for a subclass of unimodal distributions, named as proportional unimodal distributions; symmetric unimodal distributions are a special case of this subclass. These lower bounds are compared with cost deviation of the EOQ model (Figure 2). We found the lower bound to be greater than cost deviation of the EOQ model. This conclusively demonstrates that the newsboy model is more sensitive than the EOQ model. This observation is in sync with that of Khanra & Soman (2013).

Our demonstration in Figure 2 also suggests that cost deviation increases with the ratio of demand limits (r), i.e., error in ordering decision in a high r scenario (low demand range) costs more than same amount of error in a low r scenario (high demand range). Khanra & Soman (2013) had similar observation with normal demand distribution.

We generalized our results further by showing that the sensitivity behaviour of the newsboy model does not change when modelled as discrete problem. A novel approach is taken for studying the discrete case. We do not perform sensitivity analysis of the discrete case; we established a continuous equivalence of the discrete model and shown that the resultant approximation works well. This way, results for the continuous case are valid for the discrete case too.

Our study addresses the issue of magnitude of cost deviation in a broad setting. Still, much more can be done to better our understanding of sensitivity of the newsboy model and benefit from it. Since the newsboy model is sensitive to erroneous ordering decisions, it is important to limit order quantity deviation to low magnitude. However, little is known about order quantity deviation; the presence of multiple parameters makes this study complex. A thorough investigation into parameter importance can be helpful in limiting the order quantity deviation. The issue of distribution of cost and order quantity deviations remains unattended too.

Acknowledgements

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Appendix A

We prove Lemma 1 and Corollary 1 for an arbitrary $F \in \mathcal{UD}_{a,b,c,\theta}$.

Due to convexity of $F(x)$ in $[a, c]$, $F(\lambda a+(1-\lambda)c) \leq \lambda F(a)+(1-\lambda)F(c) = (1-\lambda)\theta \; \forall\lambda \in (0, 1)$. Denoting $\lambda a+(1-\lambda)c = x$, i.e., $\lambda = (c-x)/(c-a)$, $F(x) \leq \{(x-a)/(b-a)\}\theta = F_0(x) \; \forall x \in (a, c)$. Similarly, due to concavity of $F(x)$ in $[c, b]$, $F(\lambda c + (1 - \lambda)b) \geq \lambda F(c) + (1 - \lambda)F(b) = 1 - \lambda(1 - \theta) \; \forall\lambda \in [0, 1)$. Denoting $\lambda c + (1 - \lambda)b = x$, i.e., $\lambda = (b - x)/(b - c)$, $F(x) \geq 1 - \{(b - x)/(b - c)\}(1 - \theta) = F_0(x) \; \forall x \in [c, b)$. Putting these pieces together, we get Lemma 1.

If $cf < \theta$, $Q^* < c$. By Lemma 1, $F(x) \leq F_0(x) \; \forall x \in [a, c)$. Assuming contradiction, let $Q^* < Q_0^*$ for some $cf < \theta$. Then $cf = F(Q^*) \leq F_0(Q^*) < F_0(Q_0^*) = cf$, which is impossible. Hence, $Q^* \geq Q_0^*$ if $cf < \theta$. If $cf \geq \theta$, $Q^* \geq c$. By Lemma 1, $F(x) \geq F_0(x) \; \forall x \in [c, b]$. Assuming contradiction, let $Q^* > Q_0^*$ for some $cf \geq \theta$. Then $cf = F_0(Q_0^*) \leq F(Q_0^*) < F(Q^*) = cf$, which is impossible. Hence, $Q^* \leq Q_0^*$ if $cf \geq \theta$. Putting these pieces together, we get Corollary 1.

Appendix B

We prove Proposition 1 for an arbitrary $F \in \mathcal{UD}_{a,b,c,\theta}$.

Let $cf < \theta$. Then $a < Q_0^* \leq Q^* < c$ (by Corollary 1). By Lemma 1, $F(x) \leq F_0(x) \; \forall x \in [a, c)$. $F(x) < cf \; \forall x < Q^*$. Using (4),

$$\begin{aligned} D_0 - D &= (\mu_0 - \mu)cf + (Q^* - Q_0^*)cf + \int_a^{Q_0^*} \{F_0(x) - F(x)\}\mathrm{d}x - \int_{Q_0^*}^{Q^*} F(x)\mathrm{d}x \\ &= (\mu_0 - \mu)cf + \int_a^{Q_0^*} \{F_0(x) - F(x)\}\mathrm{d}x + \int_{Q_0^*}^{Q^*} \{cf - F(x)\}\mathrm{d}x \\ &\geq (\mu_0 - \mu)cf. \end{aligned}$$

Let $cf \geq \theta$. Then $c \leq Q^* \leq Q_0^* < b$ (by Corollary 1). By Lemma 1, $F(x) \geq F_0(x) \; \forall x \in [c, b]$. $F(x) > cf \; \forall x > Q^*$. Using (4),

$$\begin{aligned} D_0 - D &= (\mu_0 - \mu)cf - (Q_0^* - Q^*)cf + \int_a^{Q_0^*} \{F_0(x) - F(x)\}\mathrm{d}x + \int_{Q^*}^{Q_0^*} F(x)\mathrm{d}x \\ &= (\mu_0 - \mu)cf + \int_a^{c} \{F_0(x) - F(x)\}\mathrm{d}x - \int_c^{Q_0^*} \{F(x) - F_0(x)\}\mathrm{d}x + \int_{Q^*}^{Q_0^*} \{F(x) - cf\}\mathrm{d}x \\ &\geq (\mu_0 - \mu)cf + \int_a^{c} \{F_0(x) - F(x)\}\mathrm{d}x - \int_c^{b} \{F(x) - F_0(x)\}\mathrm{d}x. \end{aligned}$$

Now, $\mu = \int_a^b xf(x)\mathrm{d}x = bF(b) - aF(a) - \int_a^b F(x)\mathrm{d}x = b - \int_a^b F(x)\mathrm{d}x.$

$$\Rightarrow \mu_0 - \mu = \int_a^b \{F(x) - F_0(x)\}\mathrm{d}x = \int_c^b \{F(x) - F_0(x)\}\mathrm{d}x - \int_a^c \{F_0(x) - F(x)\}\mathrm{d}x.$$

Let $\int_c^b \{F(x) - F_0(x)\}\mathrm{d}x = B$ and $\int_a^c \{F_0(x) - F(x)\}\mathrm{d}x = A$. Then

$$D_0 - D \geq \begin{cases} cf(B - A) & \text{if } cf < \theta \\ -(1 - cf)(B - A) & \text{if } cf \geq \theta. \end{cases}$$

Figure 5 depicts A and B. Clearly, $A \geq \min\{A : F \in \mathcal{UD}_{a,b,c,\theta}\} = A_0 = 0$. However, maximum does not exist for $\{A : F \in \mathcal{UD}_{a,b,c,\theta}\}$. Hence, $A < \sup\{A : F \in \mathcal{UD}_{a,b,c,\theta}\} = (c-a)\theta/2$. Similarly, $0 \leq B < (b - c)(1 - \theta)/2$. Then $-(c - a)\theta/2 < B - A < (b - c)(1 - \theta)/2$.

$$\Rightarrow \ D < \begin{cases} D_0 + \frac{1}{2}cf(c - a)\theta & \text{if } cf < \theta \\ D_0 + \frac{1}{2}(1 - cf)(b - c)(1 - \theta) & \text{if } cf \geq \theta. \end{cases}$$

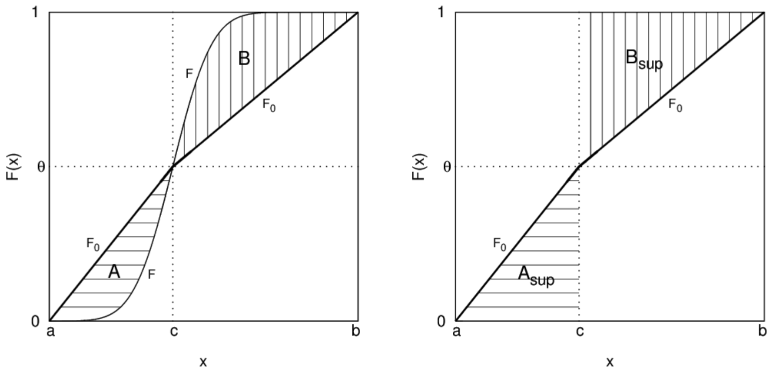


Figure 5: A and B for $F \in \mathcal{UD}_{a,b,c,\theta}$

Appendix C

We derive (6) in two parts: i) when $cf < \theta$ and ii) when $cf \geq \theta$.

Since $\mu = \int_a^b xf(x)\mathrm{d}x = bF(b) - aF(a) - \int_a^b F(x)\mathrm{d}x = b - \int_a^b F(x)\mathrm{d}x$, (4) can be rewritten as

$$\begin{aligned} D_0 &= \left[b - \int_a^b F_0(x)\mathrm{d}x - Q_0^*\right] cf + \int_a^{Q_0^*} F_0(x)\mathrm{d}x \\ &= (1 - cf)\int_a^{Q_0^*} F_0(x)\mathrm{d}x + cf\int_{Q_0^*}^b \{1 - F_0(x)\}\mathrm{d}x. \end{aligned} \tag{17}$$

Part-i) If $cf < \theta$, $Q_0^* < c$. Putting $F_0(x)$ of (1) into (17),

$$
\begin{aligned}
D_0 &= (1-cf)\int_a^{Q_0^*}\frac{x-a}{c-a}\theta \mathrm{d}x + cf\int_{Q_0^*}^{c}\left(1-\frac{x-a}{c-a}\theta\right)\mathrm{d}x + cf\int_c^b \frac{b-x}{b-c}(1-\theta)\mathrm{d}x \\
&= \frac{(1-cf)\theta}{c-a}\frac{(Q_0^*-a)^2}{2} + cf(c-Q_0^*) - \frac{cf\theta}{c-a}\frac{(c-a)^2-(Q_0^*-a)^2}{2} + \frac{cf(1-\theta)}{b-c}\frac{(b-c)^2}{2}.
\end{aligned}
$$

Putting Q_0^* of (5) into the above expression,

$$
\begin{aligned}
D_0 &= \frac{cf^2(1-cf)}{2\theta}(c-a) + \frac{cf(\theta-cf)}{\theta}(c-a) - \frac{cf(\theta^2-cf^2)}{2\theta}(c-a) + \frac{cf(1-\theta)}{2}(b-c) \\
&= \frac{cf}{2}\left[\frac{c-a}{\theta}\left\{cf(1-cf)+2(\theta-cf)-(\theta^2-cf^2)\right\} + (1-\theta)(b-c)\right] \\
&= \frac{cf}{2}\left[\frac{(1-cf)-(1-\theta)^2}{\theta}(c-a) + (1-\theta)(b-c)\right].
\end{aligned}
$$

Part-ii) If $cf \geq \theta$, $Q_0^* \geq c$. Putting $F_0(x)$ of (1) into (17),

$$
\begin{aligned}
D_0 &= (1-cf)\int_a^{c}\frac{x-a}{c-a}\theta \mathrm{d}x + (1-cf)\int_{c}^{Q_0^*}\left[1-\frac{b-x}{b-c}(1-\theta)\right]\mathrm{d}x + cf\int_{Q_0^*}^b \frac{b-x}{b-c}(1-\theta)\mathrm{d}x \\
&= \frac{(1-cf)\theta}{c-a}\frac{(c-a)^2}{2} + (1-cf)\left[(Q_0^*-c) - \frac{1-\theta}{b-c}\frac{(b-c)^2-(b-Q_0^*)^2}{2}\right] + \frac{cf(1-\theta)}{b-c}\frac{(b-Q_0^*)^2}{2}.
\end{aligned}
$$

Putting Q_0^* of (5) into the above expression,

$$
\begin{aligned}
D_0 &= \frac{1-cf}{2}\left[\theta(c-a) + \frac{2(cf-\theta)}{1-\theta}(b-c) - \frac{(1-\theta)^2-(1-cf)^2}{1-\theta}(b-c) + \frac{cf(1-cf)}{1-\theta}(b-c)\right] \\
&= \frac{1-cf}{2}\left[\theta(c-a) + \frac{b-c}{1-\theta}\left\{2(cf-\theta)-(1-\theta)^2+(1-cf)^2+cf(1-cf)\right\}\right] \\
&= \frac{1-cf}{2}\left[\theta(c-a) + \frac{cf-\theta^2}{1-\theta}(b-c)\right].
\end{aligned}
$$

Appendix D

We prove Proposition 2 for an arbitrary $F \in \mathcal{PUD}_{a,b,c}$.

For proportional unimodal demand, $\theta = (c-a)/(b-a)$. Since $\mathcal{PUD}_{a,b,c} \subset \mathcal{UD}_{a,b,c,(c-a)/(b-a)}$, using the arguments of Appendix B (the proof of Proposition 1),

$$
D_0 - D \geq \begin{cases} cf(B-A) & \text{if } cf < (c-a)/(b-a) \\ -(1-cf)(B-A) & \text{if } cf \geq (c-a)/(b-a). \end{cases}
$$

Note that $B = \int_c^b \{F(x)-F_0(x)\}\mathrm{d}x$ and $A = \int_a^c \{F_0(x)-F(x)\}\mathrm{d}x$.

Let $(b-c)/(c-a) = k$. Then using Definition 1,

$$
\begin{aligned}
& F(c+kt) - F(c) = k\{F(c) - F(c-t)\} \;\forall t \in (0, c-a] \quad \text{and} \\
& F_0(c+kt) - F_0(c) = k\{F_0(c) - F_0(c-t)\} \;\forall t \in (0, c-a] \quad \text{as } F_0 \in \mathcal{PUD}_{a,b,c}. \\
& \Rightarrow\; F(c+kt) - F_0(c+kt) = k\{F_0(c-t) - F(c-t)\} \;\forall t \in (0, c-a]. \\
& \Rightarrow\; \int_0^{c-a} \{F(c+kt) - F_0(c+kt)\}\mathrm{d}t = k\int_0^{c-a} \{F_0(c-t) - F(c-t)\}\mathrm{d}t.
\end{aligned}
$$

Replacing $c+kt = x$ in the first integral and $c-t = y$ in the second integral,

$$\int_c^b \{F(x) - F_0(x)\}\frac{\mathrm{d}x}{k} = k\int_c^a \{F_0(y) - F(y)\}(-\mathrm{d}y) \;\Rightarrow\; B = k^2 A.$$

Following the arguments of Appendix B, $0 \le A < (c-a)^2/\{2(b-a)\} = A_{sup}$. Here $B - A = (k^2-1)A$; hence, $(k^2-1)^- A_{sup} \le B - A \le (k^2-1)^+ A_{sup}$. Now $(k^2-1)A_{sup} = (a+b-2c)/2$. Then $(1/2)(a+b-2c)^- \le B - A \le (1/2)(a+b-2c)^+$.

$$\Rightarrow\; D \le \begin{cases} D_0 - \frac{1}{2}cf(a+b-2c)^- & \text{if } cf < \frac{c-a}{b-a} \\ D_0 + \frac{1}{2}(1-cf)(a+b-2c)^+ & \text{if } cf \ge \frac{c-a}{b-a}. \end{cases}$$

Appendix E

For proportional unimodal distributions, $\theta = (c-a)/(b-a)$. Then Proposition 1 takes the following form

$$D < \begin{cases} D_0 + \frac{1}{2}cf(c-a)^2/(b-a) & \text{if } cf < \frac{c-a}{b-a} \\ D_0 + \frac{1}{2}(1-cf)(b-c)^2/(b-a) & \text{if } cf \ge \frac{c-a}{b-a} \end{cases}$$

Let us denote right-hand side of the above expressions by D_{ud}. Similarly, we denote right-hand side of expressions in Proposition 2 by D_{pud}. To exhibit superiority of upper bound of D in Proposition 2 over that in Proposition 1, we need to show that $D_{pud} < D_{ud}$.

$$D_{ud} - D_{pud} < \begin{cases} \frac{1}{2}cf(c-a)^2/(b-a) + \frac{1}{2}cf(a+b-2c)^- & \text{if } cf < \frac{c-a}{b-a} \\ \frac{1}{2}(1-cf)(b-c)^2/(b-a) - \frac{1}{2}(1-cf)(a+b-2c)^+ & \text{if } cf \ge \frac{c-a}{b-a}. \end{cases}$$

If $cf < (c-a)/(b-a)$,

$$D_{ud} - D_{pud} < \begin{cases} \frac{cf}{2}\frac{(c-a)^2}{b-a} + 0 > 0 & \text{if } a+b \ge 2c \\ \frac{cf}{2}\left[\frac{(c-a)^2}{b-a} + (a+b-2c)\right] = \frac{cf}{2}\frac{(b-c)^2}{b-a} > 0 & \text{if } a+b < 2c. \end{cases}$$

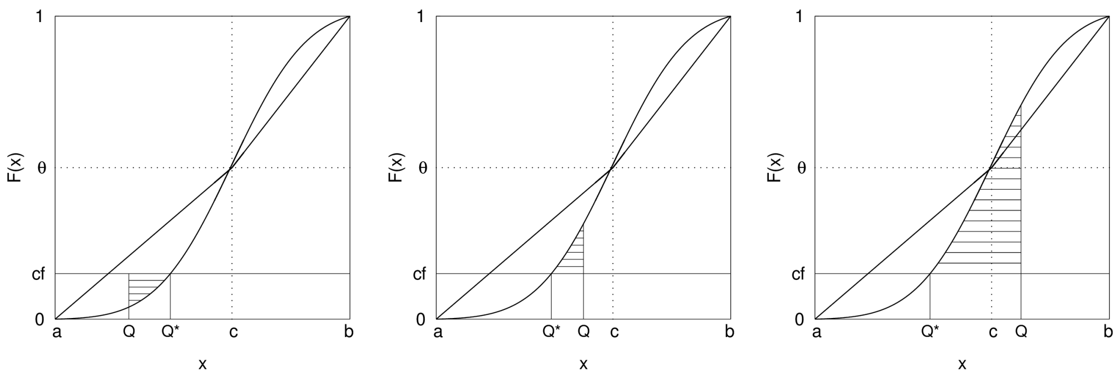
Similarly, if $cf \geq (c-a)/(b-a)$,

$$D_{ud} - D_{pud} < \begin{cases} \frac{1-cf}{2}\left[\frac{(b-c)^2}{b-a} - (a+b-2c)\right] = \frac{1-cf}{2}\frac{(c-a)^2}{b-a} > 0 & \text{if } a+b > 2c \\ \frac{1-cf}{2}\frac{(b-c)^2}{b-a} + 0 > 0 & \text{if } a+b \leq 2c. \end{cases}$$

Clearly, the bound of D in Proposition 2 is stronger than that in Proposition 1.

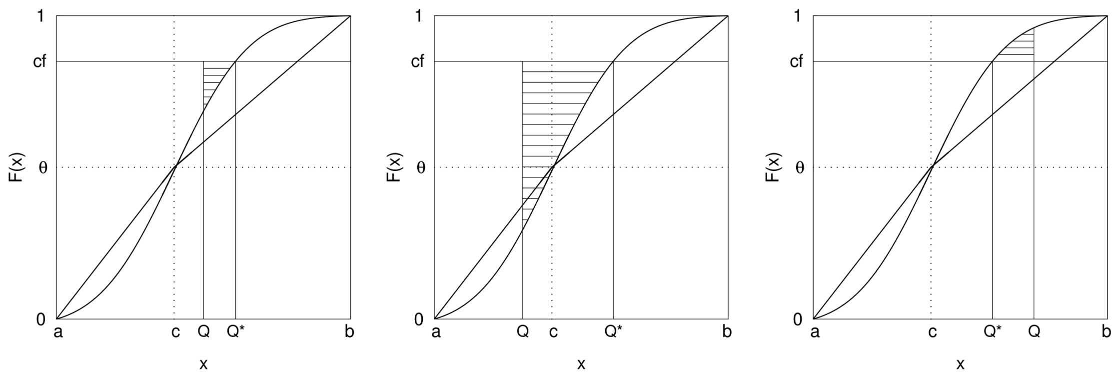
Appendix F

We prove Proposition 3 for an arbitrary $F \in \mathcal{UD}_{a,b,c,\theta}$. If $\delta_Q = 0$, the result holds vacuously. We consider the $\delta_Q \neq 0$ cases in four parts. Figure 6 and 7 show $N(\delta_Q)$ in different situations (value of cf w.r.t. θ and sign of δ_Q).



(a) $\delta_Q < 0$ (b) $\delta_Q > 0$ (c) $\delta_Q > 0$

Figure 6: $N(\delta_Q)$ when $cf < \theta$



(a) $\delta_Q < 0$ (b) $\delta_Q < 0$ (c) $\delta_Q > 0$

Figure 7: $N(\delta_Q)$ when $cf \geq \theta$

Case-IA: Let $cf < \theta$ and $\delta_Q < 0$ (Figure 6(a)). Since $cf < \theta$, by Corollary 1, $a < Q_0^* \leq Q^* < c$. Since $\delta_Q < 0$, $a \leq Q < Q^*$ as $Q \in [a, b]$.

$F(x)$ is convex in $[a, c]$. $[Q, Q^*] \subseteq [a, Q^*] \subset [a, c]$. So $F\{\lambda a + (1-\lambda)Q^*\} \leq \lambda F(a) + (1-\lambda)F(Q^*) = (1-\lambda)cf\ \forall \lambda \in [0,1]$. Putting $\lambda a + (1-\lambda)Q^* = x$, $cf - F(x) \geq \lambda cf = cf(Q^* - x)/(Q^* - a)\ \forall x \in [a, Q^*]$. Then

$$\begin{aligned} N(\delta_Q) &= \int_Q^{Q^*} \{cf - F(x)\}\mathrm{d}x \geq \int_Q^{Q^*} cf\frac{Q^* - x}{Q^* - a}\mathrm{d}x = \frac{cf(Q^*\delta_Q)^2}{2(Q^* - a)} \\ &\geq \frac{Q_0^* cf}{2}\frac{\delta_Q^2}{1 - a/Q^*} \geq \frac{Q_0^* cf}{2}\frac{\delta_Q^2}{1 - a/c} = \frac{Q_0^* cf}{2}\frac{c}{c-a}\delta_Q^2. \end{aligned}$$

Here, $\delta = \min\{\delta_Q, c/Q_0^* - 1\} = \delta_Q$. Hence, we can rewrite above expression as

$$N(\delta_Q) \geq \frac{Q_0^* cf}{2}\left[\frac{c}{c-a}\delta^2 + \frac{1-\theta}{cf}\left\{\frac{Q_0^*}{b-c}(\delta_Q^2 - \delta^2) + 2\left(\frac{\theta - cf}{1-\theta} - \frac{c - Q_0^*}{b-c}\right)(\delta_Q - \delta)\right\}\right].$$

Case-IB: Let $cf < \theta$ and $\delta_Q > 0$ (Figure 6(b) and 6(c)). Since $cf < \theta$, by Corollary 1, $a < Q_0^* \leq Q^* < c$. Since $\delta_Q > 0$, $Q^* < Q < b$ as $Q \in [a, b]$. Here

$$N(\delta_Q) = \int_{Q^*}^{\min\{Q,c\}} \{F(x) - cf\}\mathrm{d}x + \int_{\min\{Q,c\}}^{Q} \{F(x) - cf\}\mathrm{d}x.$$

Let us consider the first integrand. Due to unimodality, f is increasing in $[a, c]$. $[Q^*, \min\{Q, c\}] \subseteq [Q^*, c] \subset [a, c]$. For every $x \in [Q^*, c]$, $F(x) = F(Q^*) + \int_{Q^*}^x f(t)\mathrm{d}t \geq cf + f(Q^*)(x - Q^*)$; for the same reason, $F(Q^*) = \int_a^{Q^*} f(t)\mathrm{d}t \leq f(Q^*)(Q^* - a) \Rightarrow f(Q^*) \geq cf/(Q^* - a)$. Hence, $F(x) \geq cf + cf(x - Q^*)/(Q^* - a) \Rightarrow F(x) - cf \geq cf(x - Q^*)/(Q^* - a)\ \forall x \in [Q^*, c]$.

Let us consider the second integrand. If $Q \leq c$, the integral is zero irrespective of the integrand as the integration limits are same. If $Q > c$, $[\min\{Q, c\}, Q] \subseteq [c, b]$. By Lemma 1, $F(x) - cf \geq F_0(x) - cf = (\theta - cf) + (1-\theta)(x - c)/(b - c)\ \forall x \in [c, b]$.

Let $g_1(x) = cf(x - Q^*)/(Q^* - a)$ for $x \in [Q^*, Q]$. Let $g_2(x) = (\theta - cf) + (1-\theta)(x-c)/(b-c)$ if $Q > c$, else $(\theta - cf)$ for $x \in [\min\{Q, c\}, Q]$. Then

$$N(\delta_Q) \geq \int_{Q^*}^{\min\{Q,c\}} g_1(x)\mathrm{d}x + \int_{\min\{Q,c\}}^{Q} g_2(x)\mathrm{d}x.$$

Let $c = Q^*(1+e)$. Then $e = c/Q^* - 1 \leq c/Q_0^* - 1 = e_0$ (say). Let $\delta = \min\{\delta_Q, e_0\}$. Then $\delta \geq \min\{\delta_Q, e\} \Rightarrow Q^*(1+\delta) \geq Q^*(1 + \min\{\delta_Q, e\}) = \min\{Q, c\}$. Since g_1 is an increasing function, for every $x \in [\min\{Q, c\}, Q^*(1+\delta)]$, $g_1(x) \leq g_1(Q^*(1+\delta)) \leq g_1(Q^*(1+e_0)) = cf(c - Q_0^*)/(Q_0^* - aQ_0^*/Q^*) \leq cf(c - Q_0^*)/(Q_0^* - a) = \theta - cf \leq g_2(x)$.

$$\Rightarrow N(\delta_Q) \geq \int_{Q^*}^{Q^*(1+\delta)} g_1(x)\mathrm{d}x + \int_{Q^*(1+\delta)}^{Q} g_2(x)\mathrm{d}x.$$

$$\begin{aligned}
\int_{Q^*}^{Q^*(1+\delta)} g_1(x)\mathrm{d}x &= \frac{cf(Q^*\delta)^2}{2(Q^*-a)} \geq \frac{Q_0^*cf}{2}\frac{\delta^2}{1-a/Q^*} \geq \frac{Q_0^*cf}{2}\frac{c}{c-a}\delta^2. \\
\int_{Q^*(1+\delta)}^{Q} g_2(x)\mathrm{d}x &= \frac{Q^*(\delta_Q-\delta)}{2(b-c)}\{Q^*(\delta_Q+\delta)-2(c-Q^*)\}(1-\theta)+(\theta-cf)Q^*(\delta_Q-\delta) \\
&\geq \frac{Q_0^*}{2}\left\{\frac{Q^*(\delta_Q+\delta)-2(c-Q^*)}{b-c}(1-\theta)+2(\theta-cf)\right\}(\delta_Q-\delta) \\
&\geq \frac{Q_0^*}{2}\left\{\frac{Q_0^*(\delta_Q+\delta)-2(c-Q_0^*)}{b-c}(1-\theta)+2(\theta-cf)\right\}(\delta_Q-\delta) \\
&= \frac{Q_0^*}{2}(1-\theta)\left\{\frac{Q_0^*}{b-c}(\delta_Q^2-\delta^2)+2\left(\frac{\theta-cf}{1-\theta}-\frac{c-Q_0^*}{b-c}\right)(\delta_Q-\delta)\right\}.
\end{aligned}$$

$$\Rightarrow N(\delta_Q) \geq \frac{Q_0^*cf}{2}\left[\frac{c}{c-a}\delta^2+\frac{1-\theta}{cf}\left\{\frac{Q_0^*}{b-c}(\delta_Q^2-\delta^2)+2\left(\frac{\theta-cf}{1-\theta}-\frac{c-Q_0^*}{b-c}\right)(\delta_Q-\delta)\right\}\right].$$

Case-IIA: Let $cf \geq \theta$ and $\delta_Q < 0$ (Figure 7(a) and 7(b)). Since $cf \geq \theta$, by Corollary 1, $c \leq Q^* \leq Q_0^* < b$. Since $\delta_Q < 0$, $a \leq Q < Q^*$ as $Q \in [a, b]$. Here

$$N(\delta_Q) = \int_{Q}^{\max\{Q,c\}}\{cf - F(x)\}\mathrm{d}x + \int_{\max\{Q,c\}}^{Q^*}\{cf - F(x)\}\mathrm{d}x.$$

Let us consider the first integrand. If $Q \geq c$, the integral is zero irrespective of the integrand. If $Q < c$, $[Q, \max\{Q, c\}] \subseteq [a, c]$. By Lemma 1, $cf - F(x) \geq cf - F_0(x) = (cf - \theta) + \theta(c - x)/(c - a)\ \forall x \in [a, c]$.

Let us consider the second integrand. Due to unimodality, f is decreasing in $[c, b]$. $[\max\{Q, c\}, Q^*] \subseteq [c, Q^*] \subset [c, b]$. For every $x \in [c, Q^*]$, $F(x) = F(Q^*) - \int_x^{Q^*} f(t)\mathrm{d}t \leq cf - f(Q^*)(Q^* - x)$. For the same reason, $F(Q^*) = F(b) - \int_{Q^*}^{b} f(t)\mathrm{d}t \geq 1 - f(Q^*)(b - Q^*) \Rightarrow f(Q^*) \geq (1 - cf)/(b - Q^*)$. Hence, $F(x) \leq cf - (1 - cf)(Q^* - x)/(b - Q^*) \Rightarrow cf - F(x) \geq (1 - cf)(Q^* - x)/(b - Q^*)\ \forall x \in [c, Q^*]$.

Let $g_1(x) = (cf - \theta) + \theta(c - x)/(c - a)$ if $Q < c$, else $(cf - \theta)$ for $x \in [Q, \max\{Q, c\}]$. Let $g_2(x) = (1 - cf)(Q^* - x)/(b - Q^*)$ for $x \in [Q, Q^*]$. Then

$$N(\delta_Q) \geq \int_{Q}^{\max\{Q,c\}} g_1(x)\mathrm{d}x + \int_{\max\{Q,c\}}^{Q^*} g_2(x)\mathrm{d}x.$$

Let $c = Q^*(1 + e)$. Then $e = c/Q^* - 1 \geq c/Q_0^* - 1 = e_0$ (say). Let $\delta = \max\{\delta_Q, e_0\}$. Then $\delta \leq \max\{\delta_Q, e\} \Rightarrow Q^*(1 + \delta) \leq Q^*(1 + \max\{\delta_Q, e\}) = \max\{Q, c\}$. Since g_2 is a decreasing function, for every $x \in [Q^*(1 + \delta_1), \max\{Q, c\}]$, $g_2(x) \leq g_2(Q^*(1 + \delta)) \leq g_2(Q^*(1 + e_0)) = (1 - cf)(Q_0^* - c)/(bQ_0^*/Q^* - Q_0^*) \leq (1 - cf)(Q_0^* - c)/(b - Q_0^*) = cf - \theta \leq g_1(x)$.

$$\Rightarrow N(\delta_Q) \geq \int_{Q}^{Q^*(1+\delta)} g_1(x)\mathrm{d}x + \int_{Q^*(1+\delta)}^{Q^*} g_2(x)\mathrm{d}x.$$

$$\begin{aligned}
\int_{Q}^{Q^*(1+\delta)} g_1(x)\mathrm{d}x &= \frac{Q^*(\delta-\delta_Q)}{2(c-a)}\left\{-Q^*(\delta+\delta_Q)-2(Q^*-c)\right\}\theta+(cf-\theta)Q^*(\delta-\delta_Q)\\
&\geq \frac{c}{2}\left\{\frac{-Q^*(\delta+\delta_Q)-2(Q^*-c)}{c-a}\theta+2(cf-\theta)\right\}(\delta-\delta_Q)\\
&\geq \frac{c}{2}\left\{\frac{-Q_0^*(\delta+\delta_Q)-2(Q_0^*-c)}{c-a}\theta+2(cf-\theta)\right\}(\delta-\delta_Q)\\
&= \frac{c}{2}\theta\left\{\frac{Q_0^*}{c-a}(\delta_Q^2-\delta^2)+2\left(\frac{cf-\theta}{\theta}-\frac{Q_0^*-c}{c-a}\right)(\delta-\delta_Q)\right\}.\\
\int_{Q^*(1+\delta)}^{Q^*} g_2(x)\mathrm{d}x &= \frac{(1-cf)(Q^*\delta)^2}{2(b-Q^*)}\geq\frac{c(1-cf)}{2}\frac{\delta^2}{b/Q^*-1}\geq\frac{c(1-cf)}{2}\frac{c}{b-c}\delta^2.
\end{aligned}$$

$$\Rightarrow N(\delta_Q)\geq\frac{c(1-cf)}{2}\left[\frac{c}{b-c}\delta^2+\frac{\theta}{1-cf}\left\{\frac{Q_0^*}{c-a}(\delta_Q^2-\delta^2)+2\left(\frac{cf-\theta}{\theta}-\frac{Q_0^*-c}{c-a}\right)(\delta-\delta_Q)\right\}\right].$$

Case-IIB: Let $cf\geq\theta$ and $\delta_Q>0$ (Figure 7(c)). Since $cf\geq\theta$, by Corollary 1, $c\leq Q^*\leq Q_0^*<b$. Since $\delta_Q>0$, $Q^*<Q\leq b$ as $Q\in[a,b]$.

$F(x)$ is concave in $[c,b]$. $[Q^*,Q]\subseteq[Q^*,b]\subseteq[c,b]$. So $F\{\lambda Q^*+(1-\lambda)b\}\geq\lambda F(Q^*)+(1-\lambda)F(b)=1-\lambda(1-cf)$ $\forall\lambda\in[0,1]$. Putting $\lambda Q^*+(1-\lambda)b=x$, $F(x)-cf\geq(1-\lambda)(1-cf)=(1-cf)(x-Q^*)/(b-Q^*)$ $\forall x\in[Q^*,b]$. Then

$$\begin{aligned}
N(\delta_Q) &= \int_{Q^*}^{Q}\{F(x)-cf\}\mathrm{d}x\geq\int_{Q^*}^{Q}(1-cf)\frac{x-Q^*}{b-Q^*}\mathrm{d}x=\frac{(1-cf)(Q^*\delta_Q)^2}{2(b-Q^*)}\\
&\geq\frac{c(1-cf)}{2}\frac{\delta_Q^2}{b/Q^*-1}\geq\frac{c(1-cf)}{2}\frac{\delta_Q^2}{b/c-1}=\frac{c(1-cf)}{2}\frac{c}{b-c}\delta_Q^2.
\end{aligned}$$

Here, $\delta=\max\{\delta_Q,c/Q_0^*-1\}=\delta_Q$. Hence, we can rewrite above expression as

$$N(\delta_Q)\geq\frac{c(1-cf)}{2}\left[\frac{c}{b-c}\delta^2+\frac{\theta}{1-cf}\left\{\frac{Q_0^*}{c-a}(\delta_Q^2-\delta^2)+2\left(\frac{cf-\theta}{\theta}-\frac{Q_0^*-c}{c-a}\right)(\delta-\delta_Q)\right\}\right].$$

Appendix G

Here, we show that if the discrete distribution is unimodal with $c\in\{a,a+1,\dots,b\}$ as the mode (or one of the modes), its continuous equivalence is unimodal with every $c_{eq}\in(c-1/2,c+1/2]$ as a mode. Due to unimodality of the discrete distribution, $p(i)$ is increasing in $\{a,a+1,\dots,c\}$ and decreasing in $\{c,c+1,\dots,b\}$. We need to show that f is increasing in $[a-1/2,c_{eq}]$ and decreasing in $[c_{eq},b+1/2]$.

Any $x\in(a-1/2,b+1/2]$ can be uniquely written as $x=(i-1/2)+y$, where $i\in\{a,a+1,\dots,b\}$ and $y\in(0,1]$. $f(x)=p(i)$ if $x=(i-1/2)+y$ for $y\in(0,1]$. Let $a-1/2<x_1(i_1,y_1)<x_2(i_2,y_2)\leq c_{eq}$. $f(x_1)=p(i_1)\leq p(i_2)=f(x_2)$ as $a\leq i_1\leq i_2\leq c$. Hence, $f(x)$ is increasing in $(a-1/2,c_{eq}]$. Similarly, let $c_{eq}\leq x_1(i_1,y_1)<x_2(i_2,y_2)\leq b+1/2$. $f(x_1)=p(i_1)\geq p(i_2)=f(x_2)$ as

$c \leq i_1 \leq i_2 \leq b$. Hence, $f(x)$ is decreasing in $[c_{eq}, b+1/2]$.

We have not considered $x = a - 1/2$ in above arguments. $f(a-1/2) = p(a) \leq f(x)\ \forall x \in (a-1/2, c_{eq}]$. Thus, $f(x)$ is increasing in $[a-1/2, c_{eq}]$ and decreasing in $[c_{eq}, b+1/2]$.

Appendix H

First, we prove Lemma 3. Let us write $E[C_{eq}(Q)]$ in the form of (13).

$$\begin{aligned}
E[C_{eq}(Q)] &= \int_{a-1/2}^{Q} c_o(Q-x)f(x)\mathrm{d}x + \int_{Q}^{b+1/2} c_u(x-Q)f(x)\mathrm{d}x \\
&= \sum_{i=a}^{Q_0-1} \int_{i-1/2}^{i+1/2} c_o(Q-x)p(i)\mathrm{d}x + \int_{Q_0-1/2}^{Q} c_o(Q-x)p(Q_0)\mathrm{d}x \\
&\quad + \int_{Q}^{Q_0+1/2} c_u(x-Q)p(Q_0)\mathrm{d}x + \sum_{i=Q_0+1}^{b} \int_{i-1/2}^{i+1/2} c_u(x-Q)p(i)\mathrm{d}x \\
&= \sum_{i=a}^{Q_0-1} c_o p(i)(Q-i) + \frac{1}{2}c_o p(Q_0)d^2 + \frac{1}{2}c_u p(Q_0)(1-d)^2 + \sum_{i=Q_0+1}^{b} c_u p(i)(i-Q).
\end{aligned}$$

Let $\Delta = E[C_{eq}(Q)] - E[C(Q_0)]$. Using (13) for the expression of $E[C(Q_0)]$,

$$\begin{aligned}
\Delta &= \sum_{i=a}^{Q_0-1} c_o p(i)(Q-Q_0) + \frac{1}{2}c_o d^2 p(Q_0) + \frac{1}{2}c_u(1-d)^2 p(Q_0) + \sum_{i=Q_0+1}^{b} c_u p(i)(Q_0-Q) \\
&= c_o(d-1/2)P(Q_0-1) + c_o\left\{(d-1/2) + \frac{1}{2}(1-d)^2\right\}p(Q_0) \\
&\quad + \frac{1}{2}c_u(1-d)^2 p(Q_0) + c_u(1/2-d)\{1-P(Q_0)\} \\
&= \frac{1}{2}(c_o+c_u)(1-d)^2 p(Q_0) + (d-1/2)\{(c_o+c_u)P(Q_0) - c_u\}.
\end{aligned}$$

Dividing both sides by $(c_o + c_u)$, we get the desired result.

Now we prove Corollary 3. By Lemma 2, $Q^*_{eq} \in (Q^*-1/2, Q^*+1/2]$. Let $d = Q^*_{eq} - (Q^*-1/2)$.

$$P(Q^*) = F(Q^*+1/2) = cf + \int_{Q^*_{eq}}^{Q^*+1/2} p(Q^*)\mathrm{d}x = cf + p(Q^*)(1-d).$$

Plugging in above expression of $P(Q^*)$ in Lemma 3,

$$\frac{E[C_{eq}(Q^*_{eq})] - E[C(Q^*)]}{c_o+c_u} = \frac{1}{2}(1-d)^2 p(Q^*) + (d-1/2)p(Q^*)(1-d) = \frac{1}{2}p(Q^*)d(1-d).$$

Now, $d \in (0,1] \Rightarrow d(1-d) \in (0, 1/4]$. Hence, $0 < E[C_{eq}(Q)] - E[C(Q_0)] \leq (c_o+c_u)p(Q^*)/8$.

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Additional figures

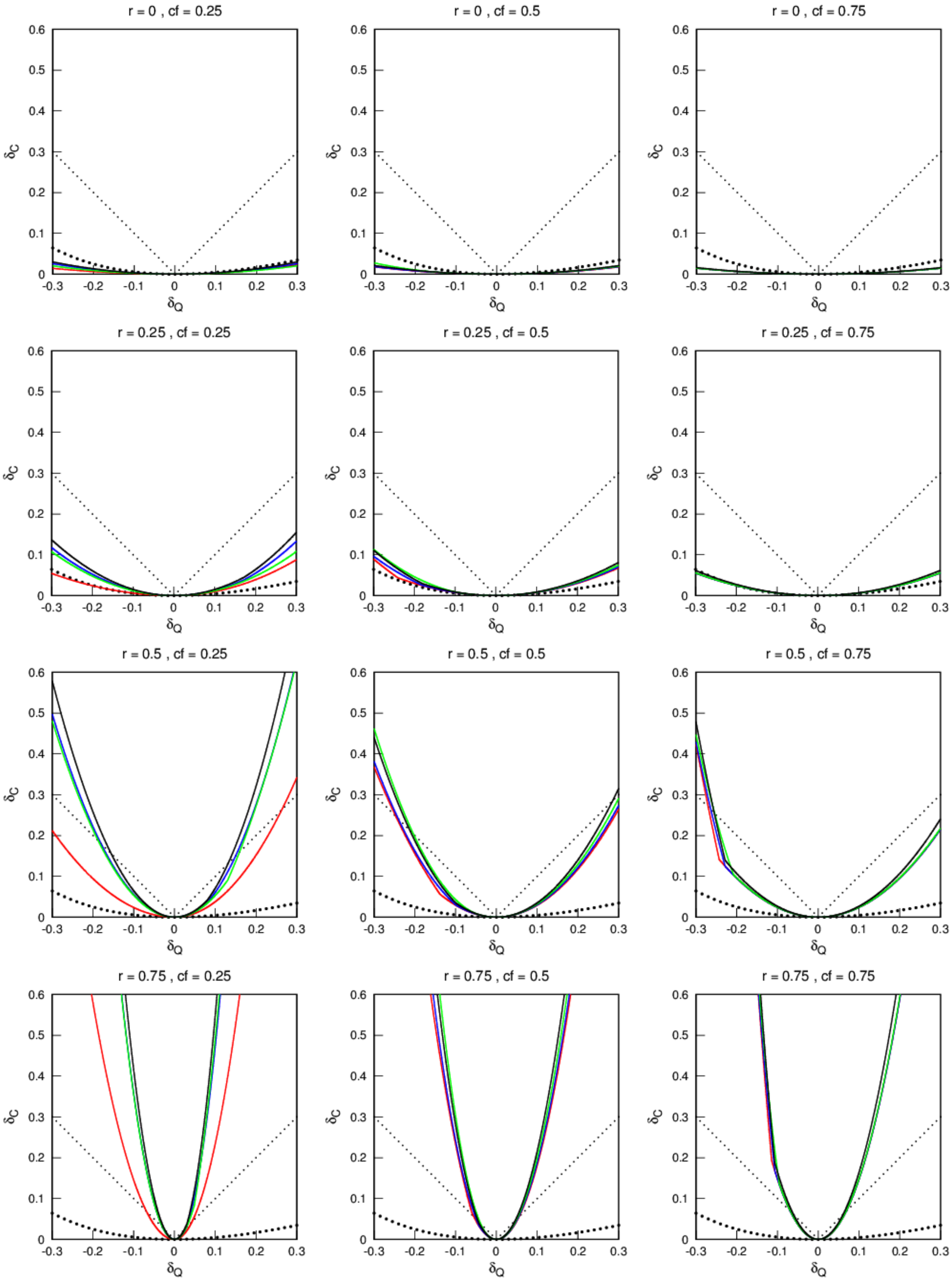

Figure 8: Lower bounds of $\delta_C(\delta_Q)$ when $m = 0.35$

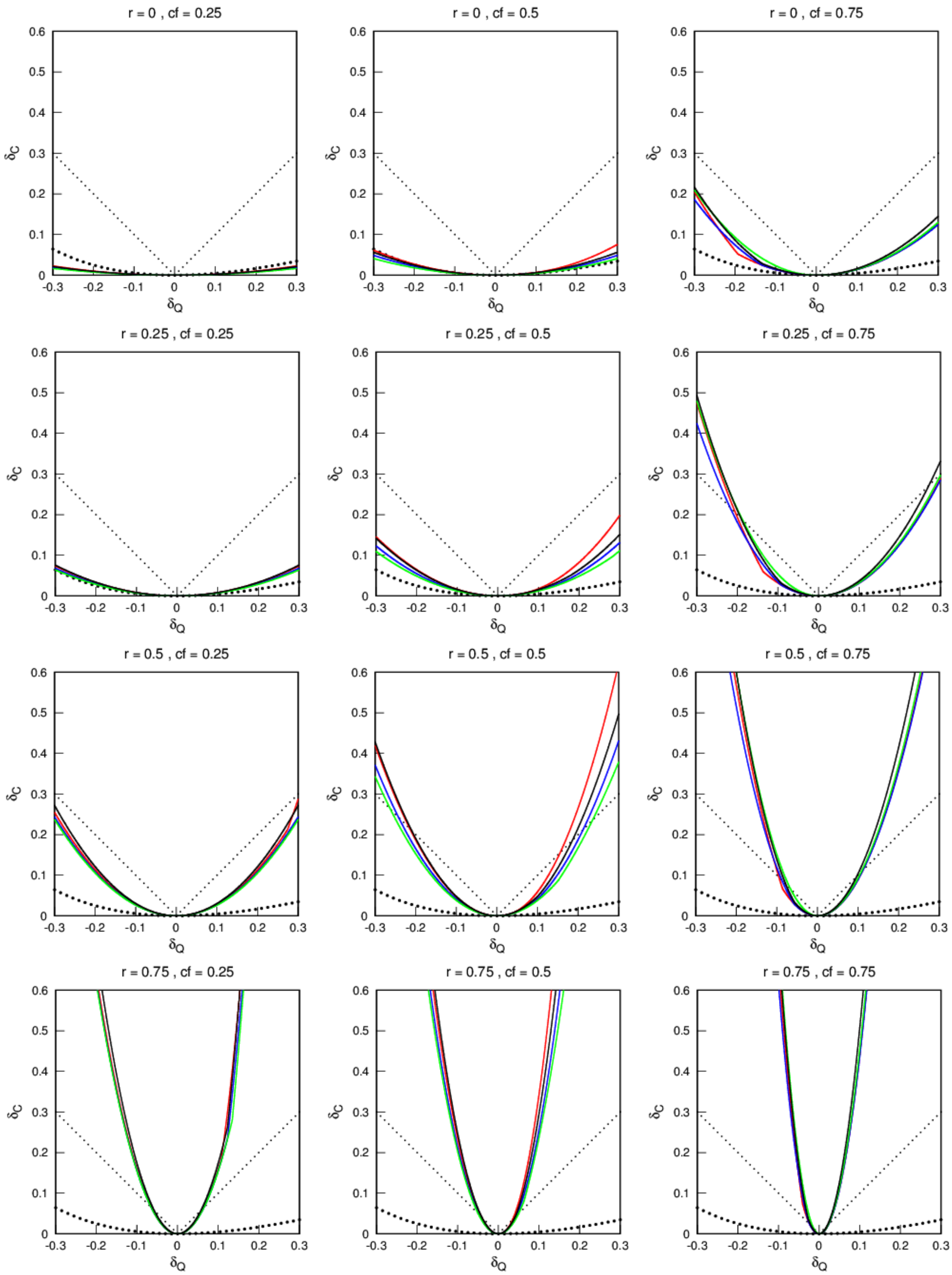

Figure 9: Lower bounds of $\delta_C(\delta_Q)$ when $m = 0.65$