INDIAN INSTITUTE OF MANAGEMENT
AHMEDABAD • INDIA

Sensitivity Analysis of the Newsboy Model

Avijit Khanra
Chetan Soman

W.P. No. 2013-09-03
September 2013

INDIAN INSTITUTE OF MANAGEMENT
AHMEDABAD-380 015
INDIA

Sensitivity analysis of the newsboy model

Avijit Khanra*[1] and Chetan Soman†[2]

[1]Doctoral Student, Indian Institute of Management Ahmedabad
[2]Faculty of Production and Quantitative Methods Area, Indian Institute of Management Ahmedabad

Abstract

Sensitivity analysis is an integral part of inventory optimization models due to uncertainty associated with estimates of model parameters. Though the newsboy problem is one of the most researched inventory problems, very little is known about its robustness. We study sensitivity of expected demand-supply mismatch cost to sub-optimal ordering decisions in the newsboy model. Conditions for symmetry (skewness) of cost deviation have been identified and magnitude of cost deviation is demonstrated for normal demand distribution. We found the newsboy model to be sensitive to sub-optimal ordering decisions, much more sensitive than the economic order quantity model.

1 Introduction

In the newsboy problem, determination of the optimal order quantity (that minimizes expected demand-supply mismatch cost) requires knowledge of cost parameters and demand distribution function. All of these may not be known correctly during decision making.

Newsboy deals with stochastic demand, whose realization takes place after the procurement decision. Hence, theoretically speaking, correct knowledge of the demand distribution (i.e., the form and associated parameters) with certainty is impossible at the time of decision making.

Unit over-stocking cost is unit purchase cost less unit salvage value (if any). In general, purchase cost is correctly known at the time of procurement decision. However, if the newsboy itself is the manufacturer and the production environment is complex (involving multiple items, etc), purchase cost (i.e., production cost specific to the concerned product) may not be known correctly. The other component of over-stocking cost, i.e., salvage value, too, may not be known correctly in many situations. Left-over inventory (if any) is sold at a secondary market (if exists) and then the remaining stock (if any) is disposed off. This process begins after the selling season is over and it involves multiple agents, thereby making the assessment of salvage value at the time of procurement decision difficult.

*avijitk@iimahd.ernet.in
†chetan@iimahd.ernet.in

Unit under-stocking cost is sum of unit profit and unit stock-out goodwill loss. Unit profit is unit selling price less unit purchase cost. We have already seen that purchase cost may not be known correctly in some situations. If the selling price is market-driven (e.g., commodity products), due to time-precedence of the procurement decision over the selling season, actual selling price may not be known correctly at the time of decision making. The other component of under-stocking cost, i.e., stock-out goodwill loss is the most elusive cost component in the newsboy model. Stock-out is reflected back as loss of future demand (of the concerned product as well as other products) through complex human behaviour, which makes the quantification of goodwill loss extremely difficult (Prichard & Eagle, 1965).

In absence of correct knowledge of one or more model parameters, estimates are used for decision making. However, forecasts are rarely correct; the first law of forecasting is that forecasts are almost always wrong (Bozarth & Handfield, 2006). The procurement decision using parameter estimates, hence, is rarely optimal (a correct decision using forecasts is a happy accident). Expected mismatch cost associated with a sub-optimal ordering decision is not the minimum. Given this unavoidable deviation of operational decision from the optima, it is important to study the nature of deviation of expected mismatch cost from its minimum (in short, cost deviation[1]). Answer to this question, i.e., sensitivity analysis is necessary for better implementation of the model.

The remainder of the paper is organized as follows. Different elements of sensitivity analysis are identified in Section 2. Literature review and research objective are discussed in Section 3. In Section 4, an expression for cost deviation is derived using standard results of the newsboy model. Necessary and sufficient conditions for symmetry (skewness) of cost deviation are identified in Section 5. In Section 6, magnitude of cost deviation is demonstrated for normal demand distribution along with a discussion on "special place" of normal distribution in the family of symmetric unimodal distributions. A brief study of order quantity deviation along with some demonstrations is carried out in Section 7. Contributions and their implications are discussed in Section 8. Finally, we conclude in Section 9.

2 Elements of sensitivity analysis

It is necessary to identify different elements of sensitivity analysis before performing it. Let us consider a simple optimization problem: minimize $y = g(x, p)$ in $x \in X$, where p represents model parameter(s). Let us assume that $x^* = h(p)$ minimizes y. Newsboy model is very similar to this problem. In fact, many inventory optimization models including the economic order quantity (EOQ) model are like this. Now, p may not be known correctly; let $\widehat{p}$ be its estimate. Then the operational decision (probably sub-optimal) is $\widehat{x}^* = h(\widehat{p})$. Let $\delta_p, \delta_x, \delta_y$ be the deviations of p, x, y from their respective true/optimal values. In sensitivity analysis, we study δ_y.

[1]Similarly, deviation of order quantity from its optimum is termed as order quantity deviation.

It can be noticed that parameter estimation error influences the objective function via the decision variable, not "directly". Sensitivity analysis of this this kind of models have two distinct parts: i) sensitivity of objective function to sub-optimal decision (δ_y to δ_x) and ii) sensitivity of decision variable to parameter estimation error (δ_x to δ_p). Combining these two parts, sensitivity of objective function to parameter estimation error (δ_y to δ_p) can be constructed. In the presence of multiple parameters, the question of parameter importance (in influencing the objective function) arises too.

We need to identify what qualifies as component of "sensitivity of δ_y to δ_x". Four components are listed here: i) direction of deviation, ii) symmetry (skewness) of deviation, iii) magnitude of deviation, and iv) distribution of deviation. In the literature review, we explore studies on these four and other relevant components.

Scenario analysis: Before proceeding to the literature review, it is important to distinguish sensitivity analysis from scenario analysis, a similar post-optimality analysis. In scenario analysis, we study the nature of change in objective function and decision variable as scenario changes, i.e., parameter value changes. The issue of sub-optimal decision does not arise here. Let the new parameter value be p'. Let the deviations from old values be $\delta_{p'}, \delta_{x'}, \delta_{y'}$. If $\widehat{p} = p'$, $\widehat{x}^* = h(\widehat{p}) = h(p') = x'^*$; however, $\widehat{y}^* = g(\widehat{x}^*, p) \neq g(x'^*, p') = y'^*$ as $p \neq p'$. Then sensitivity of δ_x to δ_p is same as sensitivity of $\delta_{x'}$ to $\delta_{p'}$, but other analogous parts differ from each other. Hence, scenario analysis results regarding the decision variable are valid for sensitivity analysis; however, the same is not true for the objective function. Scenario analysis studies are included in literature review for its similarity with sensitivity analysis.

3 Literature review

Sensitivity analysis of some important inventory models are available in literature. EOQ model is one of the most popular inventory models; one reason for its popularity is its robustness (less sensitiveness). Sensitivity of cost deviation to sub-optimal order quantity in the EOQ model can be found in standard textbooks on operations management or inventory management (e.g., Nahmias, 2001). Cost deviation is left-skewed and relatively insensitive to order quantity deviation; it grows with the second root of order quantity deviation. Lowe & Schwarz (1983) and Dobson (1988) studied sensitivity of cost deviation to parameter estimation error in the EOQ model. They observed insensitivity of cost deviation to parameter estimation error; cost deviation grows with the fourth root of parameter estimation error. Borgonovo & Peccati (2007) measured parameter importance in the EOQ model. Inventory holding cost is the most influential parameter if parameter estimation errors are similar, while the other two parameters (demand rate and ordering cost) are of equal importance.

Some sensitivity analysis results are available for popular stochastic inventory models. Zheng (1992) compared stochastic (r,Q) inventory system with its deterministic counterpart (the EOQ model) and found that the magnitude of cost deviation is less than that of the EOQ model (which is quite robust itself). F. Chen & Zheng (1997) found similar result for stochastic (s,S) inventory system with exponential demand. A surprise omission from this list is the newsboy model, one of the most researched inventory management problems. There are some good reviews on the newsboy problem (Porteus, 1990; Silver, Pyke, & Peterson, 1998; Khouja, 1999; Qin, Wang, Vakharia, Chen, & Seref, 2011; Choi, 2012); but, none mention a single focused study on sensitivity analysis.

Direction of deviation of expected mismatch cost and order quantity can be understood easily for most cases by having a careful look into the standard results of the newsboy model (available in good textbooks, e.g., Silver et al., 1998). Since expected mismatch cost is convex in order quantity, expected mismatch cost always increases whenever order quantity deviates from its optima (irrespective of the cause for deviation). Order quantity increases with over-estimation of under-stocking cost and decreases with its under-estimation. This relation is exactly opposite for over-stocking cost. Order quantity is increasing in mean demand. Impact of demand variance on order quantity is not so straight forward.

Using mean preserving transformation, Gerchak & Mossman (1992) showed that order quantity is increasing in demand variability if $cf > F(\mu)$; the relation is opposite if $cf < F(\mu)$, where cf, μ, F represent critical factor, mean demand, and demand distribution function. They also showed that expected mismatch cost is increasing in demand variability (this is a scenario analysis result regarding the objective function, not valid for sensitivity analysis). Ridder, van der Laan, & Salomon (1998) provided counter-example to this result; they showed that expected mismatch cost can be less for more variable demand.

Scenario analysis results (mainly regarding direction of deviation) of some generalized newsboy problems (i.e., extensions of the standard problem) are available in literature. Lau & Lau (2002) considered a supply chain with newsboy type retailer and identified impact of demand variability on decision variables. Their conclusions are similar to that of Gerchak & Mossman (1992) when order quantity is the only decision variable (i.e., the retailer faces the standard newsboy problem). Eeckhoudt, Gollier, & Schlesinger (1995) studied risk-averse newsboy problem (risk-neutrality is equivalent to the standard problem) and identified influence of cost parameters on order quantity; their findings are similar to what we have already mentioned.

In the newsboy literature, there are studies that develop extensions of the standard newsboy problem and study sensitivity of expected cost (or expected profit) and order quantity to the parameter(s) that differentiate it from the standard problem, e.g., L.-H. Chen & Chen (2010) solved the multi-product budget-constrained newsboy problem with a reservation policy and studied sensitivity of expected profit and order quantity to budgeted amount. These studies, being contextually different, do not add much to our understanding of sensitivity of the standard newsboy model.

Based on the above literature review, the lack of focused studies on sensitivity of the newsboy model is visible. Our understanding is limited to the knowledge of directions of cost and order quantity deviations. The questions regarding symmetry, magnitude, and distribution of cost and order quantity deviations remain largely unanswered. The issue of parameter importance has not been studied either. In this work, we study symmetry and magnitude of cost deviation as order quantity deviates from its optimal. We briefly touch the topic of order quantity deviation to reconfirm the necessity to study cost deviation.

Expected profit maximization objective is equally popular in newsboy set-up. Yet we study sensitivity of cost deviation because number of parameters is less in the cost minimization formulation of the newsboy model. Sensitivity of expected profit to sub-optimal order quantity can be easily obtained from sensitivity of expected cost (see Appendix A).

4 Problem formulation

4.1 Notations and assumptions

a	Lower limit of demand. $a \geq 0$.
b	Upper limit of demand. $a < b < \infty$.
r	Ratio of demand limits. $r = a/b$, $r \in [0, 1)$.
$F()$	Distribution function of stochastic demand. $F(a) = 0$ and $F(b) = 1$.
$f()$	Density function associated with F. Mere existence of f is assumed; hence, f can be discontinuous at countable number of points in $[a, b]$. We also assume existence of F^{-1}, i.e., $f(x) > 0$ for almost all $x \in (a, b)$.
μ	Mean demand.
σ	Standard deviation of demand.
c_v	Coefficient of variation. $c_v = \sigma/\mu$, $c_v > 0$.
c_u	Unit under-stocking cost. $c_u > 0$.
c_o	Unit over-stocking cost. $c_o > 0$.
cf	Critical factor. $cf = c_u/(c_o + c_u)$, $cf \in (0, 1)$.
Q	Order quantity. $Q \geq 0$. Q^* denotes its optimal.
$C(Q)$	Demand-supply mismatch cost for a supply of Q.
δ_Q	Deviation of order quantity from its optimal.
δ_C	Deviation of expected mismatch cost from its minimum.
$\widehat{\theta}$	Estimate of θ (model parameter). θ can be μ, σ, c_u, c_o, and cf.
δ_θ	Deviation of θ from its true value. θ can be μ, σ, c_u, c_o, and cf.

4.2 The mismatch cost formulation

We follow the mismatch cost formulation of the newsboy model (Silver et al., 1998). Let X be the random demand. Then mismatch cost is given by

$$C(Q) = c_o(Q - X)^+ + c_u(X - Q)^+. \tag{1}$$

A careful look into the above expression reveals that $C(Q < a) > C(Q = a)$ and $C(Q > b) > C(Q = b)$ for every possible demand scenario. Thus, the cost minimizing order quantity is in $[a, b]$. Expected mismatch cost is given by

$$\begin{aligned} E[C(Q)] &= c_o \int_a^Q (Q - x) f(x) \mathrm{d}x + c_u \int_Q^b (x - Q) f(x) \mathrm{d}x \\ &= c_u(\mu - Q) + (c_o + c_u) \int_a^Q (Q - x) f(x) \mathrm{d}x. \end{aligned} \tag{2}$$

$E[C(Q)]$ may not be differentiable everywhere in (a, b) as f is not necessarily continuous everywhere in $[a, b]$. Still, it is fairly easy to establish strict convexity of $E[C(Q)]$ and optimality of $Q^* = F^{-1}(cf)$ (see Appendix B). Note that Q^* is unique as F^{-1} is strictly increasing. Minimum expected mismatch cost is given by

$$E[C(Q^*)] = c_u \mu - (c_o + c_u) \int_a^{Q^*} x f(x) \mathrm{d}x. \tag{3}$$

4.3 Expression for cost deviation

We use ratio-based measure for deviation. Ratio-based measure is unit-less; hence, comparison among deviations is easy. $\delta_Q = (Q - Q^*)/Q^*$ and $\delta_C = (E[C(Q)] - E[C(Q^*)])/E[C(Q^*)]$. $\delta_Q \geq -1$ as $Q \geq 0$ and $\delta_C \geq 0$ as $E[C(Q)] \geq E[C(Q^*)]$. We derive an expressions for cost deviation using the demand distribution function. Using (2) and (3),

$$\begin{aligned} E[C(Q)] - E[C(Q^*)] &= -c_u Q + (c_o + c_u) \left[\int_a^Q Q f(x) \mathrm{d}x - \int_{Q^*}^Q x f(x) \mathrm{d}x \right] \\ &= (c_o + c_u) \left[Q\{F(Q) - cf\} - \left\{ QF(Q) - Q^* cf - \int_{Q^*}^Q F(x) \mathrm{d}x \right\} \right] \\ &= (c_o + c_u) \int_{Q^*}^Q \{F(x) - cf\} \mathrm{d}x. \end{aligned}$$

$$\text{Similarly, } E[C(Q^*)] = (c_o + c_u) \left[(\mu - Q^*) cf + \int_a^{Q^*} F(x) \mathrm{d}x \right].$$

Writing $Q = Q^*(1 + \delta_Q)$,

$$\delta_C(\delta_Q) = \frac{\int_{Q^*}^{Q^*(1+\delta_Q)} \{F(x) - cf\} \mathrm{d}x}{(\mu - Q^*) cf + \int_a^{Q^*} F(x) \mathrm{d}x}. \tag{4}$$

Unlike the EOQ model, where cost deviation is independent of model parameters, $\delta_C(\delta_Q)$ in the newsboy model depends on cf and F. These dependencies, particularly the presence of demand distribution function, make the sensitivity analysis complicated. We begin our analysis with the study of symmetry of $\delta_C(\delta_Q)$ in the next section.

5 Symmetry (skewness) of cost deviation

We identify the conditions for symmetry (skewness) of $\delta_C(\delta_Q)$ w.r.t. $\delta_Q = 0$. First, we formally define symmetry, left-skewness, right-skewness, and asymmetry.

Definition 1. *Let g be a real-valued function defined on interval I. Let $x_0 \in I$ and $e_0 > 0$ such that $x_0 - e_0, x_0 + e_0 \in I$. Then we say that g is*

i) symmetric in $x_0 \pm e_0$ if $g(x_0 - e) = g(x_0 + e)$ $\forall e \in (0, e_0]$.

ii) left-skewed in $x_0 \pm e_0$ if $g(x_0 - e) > g(x_0 + e)$ $\forall e \in (0, e_0]$.

iii) right-skewed in $x_0 \pm e_0$ if $g(x_0 - e) < g(x_0 + e)$ $\forall e \in (0, e_0]$.

iv) asymmetric in $x_0 \pm e_0$ if none among the above three holds.

Above definition can be easily modified for functions defined on integer domain.

Following results connect symmetry (skewness) of $\delta_C(\delta_Q)$ in $0 \pm e_0$ with symmetry (skewness) of the demand density function, f in $Q^* \pm e_0 Q^*$. Since $\delta_Q \geq -1$, we restrict $e_0 \in (0, 1]$ so that the above definition is valid for $\delta_C(\delta_Q)$.

Proposition 1. *$\delta_C(\delta_Q)$ is symmetric in $0 \pm e_0$ if and only if the demand density function, f is symmetric almost everywhere in $Q^* \pm e_0 Q^*$.*

Proof. Denominator of $\delta_C(\delta_Q)$ in (4) is δ_Q-independent and positive. Then the numerator decides symmetry (skewness) of $\delta_C(\delta_Q)$. Let D be the denominator. Let $\Delta(e) = D\{\delta_C(0+e) - \delta_C(0-e)\}$. For every $e \in (0, e_0]$,

$$\Delta(e) = \int_{Q^*}^{Q^*(1+e)} \{F(x) - F(Q^*)\}\mathrm{d}x - \int_{Q^*}^{Q^*(1-e)} \{F(x) - F(Q^*)\}\mathrm{d}x.$$

Replacing $y = |x - Q^*|$, i.e., $x = Q^* + y$ in the first and $x = Q^* - y$ in the second integral,

$$\begin{aligned}\Delta(e) &= \int_0^{eQ^*} \{F(Q^* + y) - F(Q^*)\}\,\mathrm{d}y - \int_0^{eQ^*} \{F(Q^* - y) - F(Q^*)\}\,(-\mathrm{d}y) \\ &= \int_0^{eQ^*} \left[\int_{Q^*}^{Q^*+y} f(z)\mathrm{d}z - \int_{Q^*-y}^{Q^*} f(z)\mathrm{d}z\right] \mathrm{d}y.\end{aligned}$$

Replacing $t = |z - Q^*|$, i.e., $z = Q^* + t$ in the first and $z = Q^* - t$ in the second inner integral,

$$\begin{aligned}\Delta(e) &= \int_0^{eQ^*} \left[\int_0^{y} f(Q^* + t)\mathrm{d}t - \int_y^{0} f(Q^* - t)(-\mathrm{d}t)\right] \mathrm{d}y \\ &= \int_0^{eQ^*} \int_0^{y} \{f(Q^* + t) - f(Q^* - t)\}\mathrm{d}t\ \mathrm{d}y.\end{aligned}$$

Let f be symmetric almost everywhere in $Q^* \pm e_0 Q^*$, i.e., $f(Q^*+t) = f(Q^*-t)$ for almost all $t \in (0, e_0 Q^*]$. Then

$$\begin{aligned}
&\int_0^y \{f(Q^*+t) - f(Q^*-t)\}\mathrm{d}t = 0 \ \forall y \in (0, e_0 Q^*] \\
&\Rightarrow \int_0^{eQ^*} \int_0^y \{f(Q^*+t) - f(Q^*-t)\}\mathrm{d}t \ \mathrm{d}y = 0 \ \forall e \in (0, e_0] \\
&\Rightarrow \Delta(e) = 0 \equiv \delta_C(e) = \delta_C(-e) \ \forall e \in (0, e_0].
\end{aligned}$$

Hence, $\delta_C(\delta_Q)$ is symmetric in $0 \pm e_0$ if f is symmetric almost everywhere in $Q^* \pm e_0 Q^*$.

Now, we need to show that if f is not symmetric almost everywhere in $Q^* \pm e_0 Q^*$, $\delta_C(\delta_Q)$ is not symmetric in $0 \pm e_0$. If f is not symmetric almost everywhere in $Q^* \pm e_0 Q^*$, $f(Q^*+t) \neq f(Q^*-t)$ for uncountable points, $t \in (0, e_0 Q^*]$. Then

i) $\exists\, e_1 \in [0, e_0)$ such that f is symmetric almost everywhere in $Q^* \pm e_1 Q^*$.

ii) $\exists\, e_2 \in (e_1, e_0]$ such that either $f(Q^*+t) > f(Q^*-t)$ or $f(Q^*+t) < f(Q^*-t)$ for almost all $t \in (e_1 Q^*, e_2 Q^*]$. This is due to continuity of f almost everywhere in $[a, b]$.

For every $e \in (e_1, e_2] \subseteq (0, e_0]$,

$$\begin{aligned}
\Delta(e) &= \int_0^{e_1 Q^*} \int_0^y \{f(Q^*+t) - f(Q^*-t)\}\mathrm{d}t \ \mathrm{d}y + \int_{e_1 Q^*}^{eQ^*} \int_0^y \{f(Q^*+t) - f(Q^*-t)\}\mathrm{d}t \ \mathrm{d}y \\
&= 0 + \int_{e_1 Q^*}^{eQ^*} \left[\int_0^{e_1 Q^*} \{f(Q^*+t) - f(Q^*-t)\}\mathrm{d}t + \int_{e_1 Q^*}^{y} \{f(Q^*+t) - f(Q^*-t)\}\mathrm{d}t \right] \mathrm{d}y \\
&= \int_{e_1 Q^*}^{eQ^*} \left[0 + \int_{e_1 Q^*}^{y} \{f(Q^*+t) - f(Q^*-t)\}\mathrm{d}t \right] \mathrm{d}y \neq 0 \ \Rightarrow \ \delta_C(e) \neq \delta_C(-e).
\end{aligned}$$

Thus, $\delta_C(\delta_Q)$ is not symmetric in $0 \pm e_0$ if f is not symmetric almost everywhere in $Q^* \pm e_0 Q^*$. □

Density function of uniformly distributed demand is symmetric in $Q^* \pm e_0 Q^*$ for any Q^* (i.e., any cf) if e_0 is not large (such that $a \leq Q^*(1-e_0)$ and $Q^*(1+e_0) \leq b$). Then by the above proposition, $\delta_C(\delta_Q)$ is symmetric. The result related to skewness is somewhat different.

Proposition 2. *$\delta_C(\delta_Q)$ is left (right) skewed in $0 \pm e_0$ if the demand density function, f is left (right) skewed almost everywhere in $Q^* \pm e_0 Q^*$. Conversely, if $\delta_C(\delta_Q)$ is left (right) skewed in $0 \pm e_0$, f is left (right) skewed almost everywhere in $Q^* \pm e_0' Q^*$ for some $e_0' \in (0, e_0]$.*

Proof. Following the arguments of proposition 1, for every $e \in (0, e_0]$,

$$\Delta(e) = \int_0^{eQ^*} \int_0^y \{f(Q^*+t) - f(Q^*-t)\}\mathrm{d}t \ \mathrm{d}y.$$

Let f be left-skewed almost everywhere in $Q^* \pm e_0 Q^*$, i.e., $f(Q^*+t) < f(Q^*-t)$ for almost all $t \in (0, e_0 Q^*]$. Then

$$\begin{aligned}
&\int_0^y \{f(Q^*+t) - f(Q^*-t)\}\mathrm{d}t < 0 \ \forall y \in (0, e_0 Q^*] \\
&\Rightarrow \int_0^{eQ^*} \int_0^y \{f(Q^*+t) - f(Q^*-t)\}\mathrm{d}t \ \mathrm{d}y < 0 \ \forall e \in (0, e_0] \\
&\Rightarrow \Delta(e) < 0 \equiv \delta_C(e) < \delta_C(-e) \ \forall e \in (0, e_0].
\end{aligned}$$

Hence, $\delta_C(\delta_Q)$ is left-skewed in $0 \pm e_0$ if f is left-skewed almost everywhere in $Q^* \pm e_0 Q^*$.

Now we prove the other part of the proposition. By contradiction, let us assume that $\delta_C(\delta_Q)$ is left-skewed in $0 \pm e_0$, but $\nexists\ e_0' \in (0, e_0]$ such that f is left-skewed almost everywhere in $Q^* \pm e_0' Q^*$. Then $\exists\ e' \in (0, e_0]$ such that $f(Q^*+t) \geq f(Q^*-t)$ for almost all $t \in (0, e'Q^*]$. This is due to continuity of f almost everywhere in $[a, b]$. Then

$$\begin{aligned}
&\int_0^y \{f(Q^*+t) - f(Q^*-t)\}\mathrm{d}t \geq 0 \ \forall y \in (0, e'Q^*] \\
&\Rightarrow \int_0^{eQ^*} \int_0^y \{f(Q^*+t) - f(Q^*-t)\}\mathrm{d}t \ \mathrm{d}y \geq 0 \ \forall e \in (0, e'] \\
&\Rightarrow \Delta(e) \geq 0 \equiv \delta_C(e) \geq \delta_C(-e) \ \forall e \in (0, e'].
\end{aligned}$$

Hence, $\delta_C(\delta_Q)$ is not left-skewed in $0 \pm e'$. Since $e' \leq e_0$, $\delta_C(\delta_Q)$ is not left-skewed in $0 \pm e_0$, which is in contradiction with our assumption. Thus, if $\delta_C(\delta_Q)$ is left-skewed in $0 \pm e_0$, f is left-skewed almost everywhere in $Q^* \pm e_0' Q^*$ for some $e_0' \in (0, e_0]$.

The proposition can be proved for the case of right-skewness in a very similar manner. □

Proposition 1 states that symmetry of the demand density function, f almost everywhere in $Q^* \pm e_0 Q^*$ is both necessary and sufficient for symmetry of cost deviation, $\delta_C(\delta_Q)$ in $0 \pm e_0$. These conditions are slightly different for skewness. By proposition 2, left (right) skewness of f almost everywhere in $Q^* \pm e_0' Q^*$ for some $e_0' \in (0, e_0]$ is necessary, whereas the same in $Q^* \pm e_0 Q^*$ is sufficient for left (right) skewness of $\delta_C(\delta_Q)$ in $0 \pm e_0$. Thus, asymmetric demand density functions may lead to skewness of δ_C. Only asymmetric demand density functions can lead to asymmetry of δ_C. Proposition 1 and 2 have an interesting consequence for symmetric unimodal demand distributions.

Corollary 1. *$\delta_C(\delta_Q)$ is right-skewed if $cf < 1/2$, left-skewed if $cf > 1/2$, and symmetric if $cf = 1/2$ for symmetric unimodal (in strong sense) demand distributions.*

Proof. Here, mean and mode are same. Due to symmetry and strict unimodality, $f(x_1) < f(x_2)$ if $|\mu - x_1| > |\mu - x_2|$, $f(x_1) > f(x_2)$ if $|\mu - x_1| < |\mu - x_2|$, and $f(x_1) = f(x_2)$ if $|\mu - x_1| = |\mu - x_2|$ for any $x_1, x_2 \in [a, b]$. $F(\mu) = 1/2$ due to symmetry. Let cf, e_0 are such that $a \leq Q^*(1 - e_0)$ and $Q^*(1 + e_0) \leq b$.

If $cf < 1/2$, $Q^* = F^{-1}(cf) < \mu$. Then f is right-skewed in $Q^* \pm e_0 \cdot Q^*$, i.e., $f(Q^* - q) < f(Q^*+q)$ for every $q \in (0, e_0 \cdot Q^*]$ as $|\mu-(Q^*-q)| = |q+(\mu-Q^*)| > |q-(\mu-Q^*)| = |\mu-(Q^*+q)|$. By proposition 2, $\delta_C(\delta_Q)$ is right-skewed in $0 \pm e_0$.

If $cf > 1/2$, $Q^* = F^{-1}(cf) > \mu$. Then $f(x)$ is left-skewed in $Q^* \pm e_0 \cdot Q^*$, i.e., $f(Q^* - q) > f(Q^*+q)$ for every $q \in (0, e_0 \cdot Q^*]$ as $|\mu-(Q^*-q)| = |q-(Q^*-\mu)| < |q+(Q^*-\mu)| = |\mu-(Q^*+q)|$. By proposition 2, $\delta_C(\delta_Q)$ is left-skewed in $0 \pm e_0$.

If $cf = 1/2$, $Q^* = F^{-1}(cf) = \mu$. Then $f(Q^* - q) = f(\mu - q) = f(\mu + q) = f(Q^* + q)$ for every $q \in (0, e_0 \cdot Q^*]$, i.e., $f(x)$ is symmetric. By proposition 1, $\delta_C(\delta_Q)$ is symmetric in $0 \pm e_0$. □

Many commonly used demand distributions like normal distribution (with proper truncation) and symmetric triangular distribution are symmetric and unimodal (in strong sense). In these cases, if $cf < 1/2$, it is better to under-estimate the order quantity (than over-estimating) and if $cf > 1/2$, it is better to over-estimate the order quantity (than under-estimating). Table 1 illustrates this observation. Using (5), $\delta_C(\delta_Q)$ are calculated for different cf for normally distributed demand with $c_v = 0.25$ (no truncation). It is evident that $\delta_C(\delta_Q)$ is right-skewed for $cf = 0.25$, left-skewed for $cf = 0.75$, and symmetric for $cf = 0.5$.

Table 1: Symmetry (skewness) of $\delta_C(\delta_Q)$ for normal demand

cf	δ_Q							
	-5%	5%	-10%	10%	-15%	15%	-20%	20%
0.25	1.33	1.43	5.09	5.91	10.9	13.7	18.6	24.8
0.50	1.99	1.99	7.90	7.90	17.5	17.5	30.4	30.4
0.75	2.87	2.58	11.9	9.70	27.7	20.4	50.3	33.8

6 Normal demand distribution

Let us turn our attention to the study of magnitude of cost deviation. $\delta_C(\delta_Q)$ depends on cf and F. Since $cf \in (0, 1)$, we can capture dependence of δ_C on cf by considering different scenarios (different values of cf). On the other hand, F can be "anything"; there is no simple way to capture dependence of δ_C on F. In this situation, study of specific demand distribution(s) is a way-out. When it comes to demand modelling, normal distribution is the natural choice. It is widely used for demand modelling in the newsboy and inventory management literature (Silver et al., 1998). Another reason for choosing normal distribution is its "special place" in the family of symmetric unimodal distributions (discussed next), which allows us to generalize our findings from the study with normal distribution to the family of symmetric unimodal distributions.

6.1 Symmetric unimodal demand

We assume following three properties to hold for stochastic demand in a newsboy set-up.

I. *Boundedness*: Demand has finite non-negative lower limit and positive upper limit, i.e., F has a bounded interval support $[a, b]$, $0 \leq a < b < \infty$.

II. *Unimodality*: Demand distribution is unimodal, i.e., F is convex in $[a, c]$ and concave in $[c, b]$, where c is the mode (Gkedenko & Kolmogorov, 1954).

III. *Symmetry*: Demand density is symmetric, i.e. $f(a + x) = f(b - x)\ \forall x \in [0, b - a]$.

It is impossible to find an exception to the boundedness property in a real life situation. The unimodality property, too, holds in general; most demand distributions in the literature are unimodal. The symmetry property is somewhat restrictive; still, many real life newsboy demand can be adequately modelled by symmetric distributions. Popular distributions for demand modelling such as uniform, symmetric triangular, and symmetric truncated normal distributions satisfy above properties. The set of symmetric unimodal demand distributions in $[a, b]$ is denoted by $\mathcal{D}_{a,b}$.

We do not need to assume existence of F^{-1} separately for $F \in \mathcal{D}_{a,b}$ because unimodality ensures that $f(x) > 0\ \forall x \in (a, b)$. If $f(a' > a) = 0$, $F(a') = 0$; then a can not be the lower limit. Similarly, if $f(b' < b) = 0$, $F(b') = 1$; then b can not be the upper limit. Here, mean and mode are same due to symmetry of f. For the same reason, $\mu = (a + b)/2$ and $F(\mu) = 1/2$ for every $F \in \mathcal{D}_{a,b}$. $F \in \mathcal{D}_{a,b}$ admits following bounds (see Appendix C for a proof).

Lemma 1. *$F_0(x) < F(x) \leq F_U(x)$ if $x \in (a, \mu)$ and $F_U(x) \leq F(x) < F_0(x)$ if $x \in (\mu, b)$ for every $F \in \mathcal{D}_{a,b}$, where $F_0(x) = 0$ if $x \in [a, \mu)$, $F_0(x) = 1$ if $x \in [\mu, b]$ and F_U is the uniform distribution in $[a, b]$.*

Figure 1 demonstrates the bounds of $\mathcal{D}_{a,b}$. The thick straight line is F_U. The remaining curves correspond to symmetric truncated normal distribution for different values of coefficient of variation (in short, $F_{N(c_v)}$). $c_v = 0.01, 0.03, 0.05, 0.07, 0.1, 0.14, 0.2, 0.3$ have been considered; $c_v = 0.01$ is closest to F_0 and $c_v = 0.3$ is closest to F_U.

Any combination of a convex increasing curve connecting $(a, 0), (\mu, 0.5)$ and a concave increasing curve connecting $(\mu, 0.5), (b, 1)$ with the additional property of symmetry is a member of $\mathcal{D}_{a,b}$. $F_{N(c_v)}$ assumes different forms between the bounds of $\mathcal{D}_{a,b}$ for different values of c_v. In fact, $F_{N(c_v)}$ asymptotically approaches bounds of $\mathcal{D}_{a,b}$.

Lemma 2. *$F_{N(c_v)} \to F_U$ as $c_v \to \infty$ and $F_{N(c_v)} \to F_0$ as $c_v \to 0^+$ except around the mean, where F_U and F_0 are bounds of $F \in \mathcal{D}_{a,b}$.*

A proof of the above lemma appears in Appendix D. Due to versatility of shape of $F_{N(c_v)}$, by studying cost deviation for $F_{N(c_v)}$ with different c_v values, we can get a fair idea about the magnitude of cost deviation for the newsboy model (for $F \in \mathcal{D}_{a,b}$).

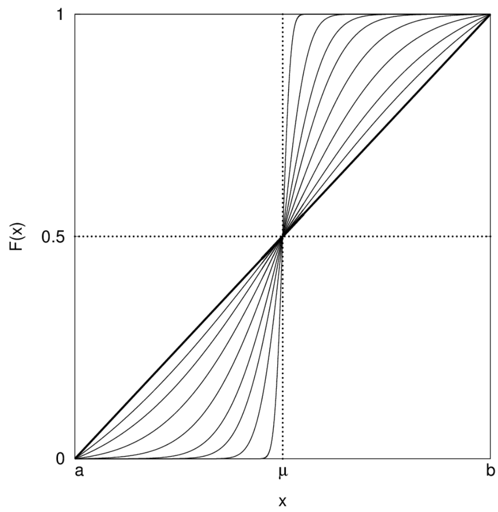

Figure 1: F_U and $F_{N(c_v)}$ in $\mathcal{D}_{a,b}$

6.2 Cost deviation for normal demand

We specify a symmetric truncated normal distribution using its lower limit (a), upper limit (b), and coefficient of variation of the underlying normal distribution ($c_v = \sigma_0/\mu_0$). We can not use (4) to evaluate cost deviation because F does not have a closed form expression for normal distribution. Instead, we use following expression (see Appendix E for details).

$$\delta_C(\delta_Q) = \frac{z\{\Phi(z) - \Phi(z^*)\} + \{\phi(z) - \phi(z^*)\}}{\phi(z^*) - \phi(-k/c_v)}, \tag{5}$$

where $z^* = \Phi^{-1}\left((1-cf)\Phi(-k/c_v) + cf\Phi(k/c_v)\right)$, $z = z^*(1+\delta_Q)+\delta_Q/c_v$, and $k = (1-r)/(1+r)$. ϕ and Φ correspond to the standard normal distribution.

One interesting observation about the above expression is that $\delta_C(\delta_Q)$ does not depend on a and b "directly"; it depends on their ratio (r). This provided an easy way to capture a wide range of scenarios by varying $r \in [0, 1)$ and $cf \in (0, 1)$. By varying c_v, the influence of demand distribution can be captured in a given scenario. However, since $c_v \in (0, \infty)$, it is impossible to capture the influence of c_v by considering finite number of values. This issue can be resolved by studying cost deviation for uniformly distributed demand because $F_{N(c_v)} \to F_U$ as $c_v \to \infty$ (by lemma 2). $\delta_C(\delta_Q)$ for uniformly distributed demand is given by following expression (see Appendix F for details).

$$\delta_C(\delta_Q) = \frac{\{r + cf(1-r)\}^2}{cf(1-cf)(1-r)^2}\delta_Q^2. \tag{6}$$

Figure 2 shows the $\delta_C(\delta_Q)$ curves in different scenarios. Each diagram corresponds to a combination of r and cf values; three values, 0.25, 0.5, and 0.75 (low, medium, and high) have been considered for both. Thin solid curves in each diagram correspond to normal distribution with different c_v values $(0.1, 0.15, 0.2, 0.3)$; a larger c_v corresponds to a flatter curve. The thick solid curve corresponds to the uniform distribution.

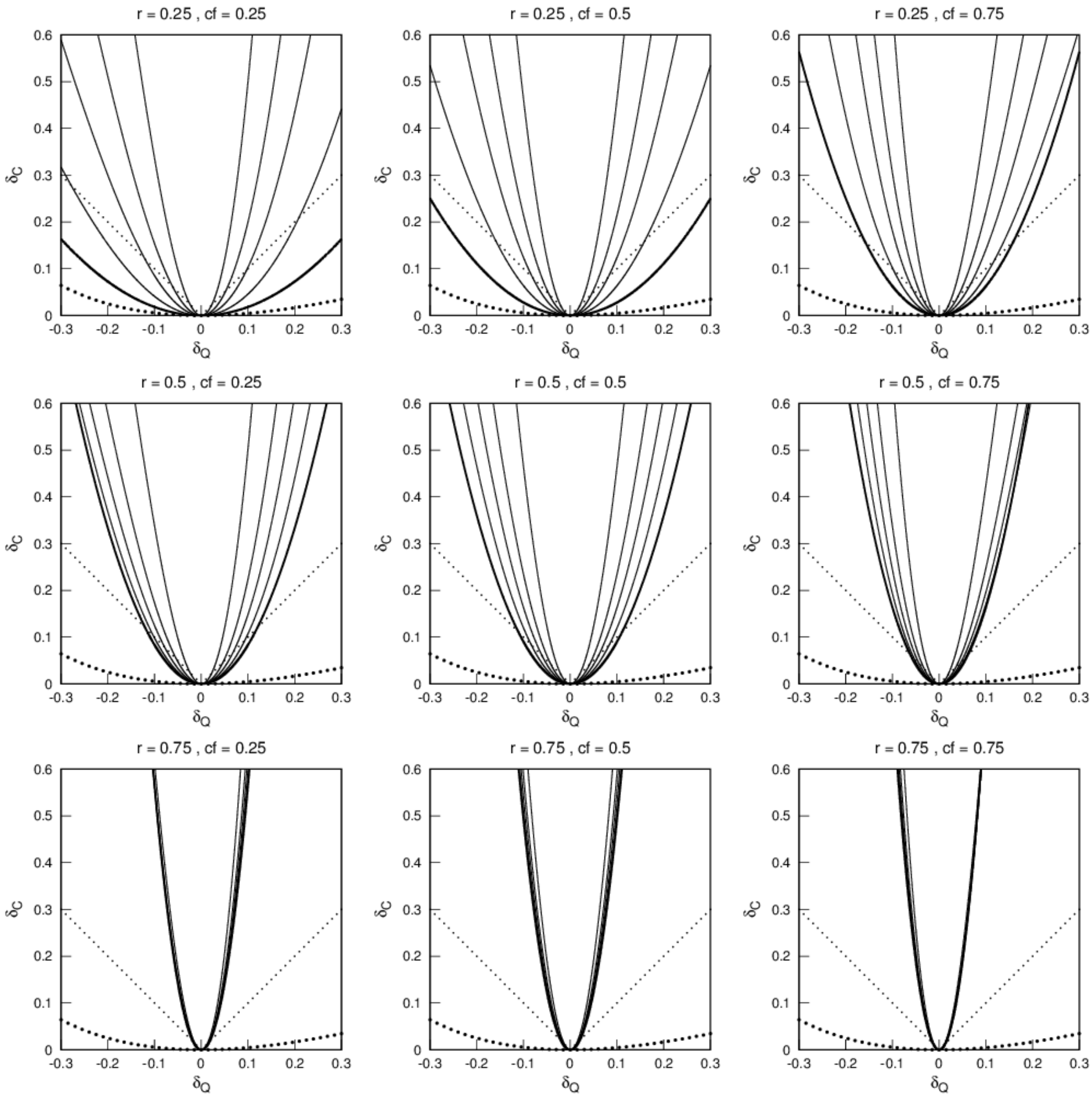

Figure 2: $\delta_C(\delta_Q)$ for normal distribution

We also indicate $\delta_C(\delta_Q)$ for the EOQ model (thick dotted curve) for easy comparison; $\delta_C(\delta_Q) = \delta_Q^2/\{2(1+\delta_Q)\}$ for the EOQ model (Nahmias, 2001). Dotted lines with slope 1 and -1 separate the error "dampening" and "amplifying" zones. If a curve (or part of it) lies below these lines, the magnitude of the output error is less than that of the input error (dampening of error). Conversely, if the curve (or part of it) lies above these lines, the output error is more in magnitude than the input error (amplification of error).

Key observations

i) $F_{N(c_v)}$ and F_U curves are steeper than the EOQ curve. Very small part of the $F_{N(c_v)}$ curves lie below the ± 1 slope lines.

ii) Steepness of $F_{N(c_v)}$ and F_U curves decrease in c_v and increase in r and cf.

Greater steepness of the $F_{N(c_v)}$ and F_U curves compared to the EOQ curve convincingly demonstrates that the newsboy model is more sensitive to sub-optimal order quantities than the EOQ model. Like EOQ curve, the ± 1 slope lines act as benchmark. Locations of the $F_{N(c_v)}$ curves imply that amplification of error occurs in many situations.

From the behaviour of the $F_{N(c_v)}$ and F_U curves as c_v and r change, we can conclude that robustness of the newsboy model deteriorates with decrease in c_v and increase in r. This behaviour can be explained by flattening of the density function due to increased c_v or decreased r. In both cases, the numerator of cost deviation, $\int_{Q^*}^{Q}\{F(x) - cf\}\mathrm{d}x = \int_{Q^*}^{Q}(Q - x)f(x)\mathrm{d}x$ decreases, thereby decreasing $\delta_C(\delta_Q)$. The effect reverses when c_v decreases or r increases.

When cf is high, Q^* is high; then $Q = Q^*(1 + \delta_Q)$ is at a greater distance from Q^* compared to a low cf case. A higher $|Q - Q^*|$ increases the numerator of cost deviation, $\int_{Q^*}^{Q}\{F(x) - cf\}\mathrm{d}x$, thereby increasing $\delta_C(\delta_Q)$. This explains behaviour of the $F_{N(c_v)}$ and F_U curves as cf changes. This behaviour is mainly due to the choice of measurement of deviation. We may not observe the same pattern if deviation is measured by change in value.

The special case of $r = 0$

In Figure 2, $r = 0.25, 0.5, 0.75$ have been considered. Theoretically, $r \in [0, 1)$. Though very high value of r is uncommon, very low value, on the other hand, is quite common, e.g., demand limits of 10 and 100 gives a r value of 0.1.

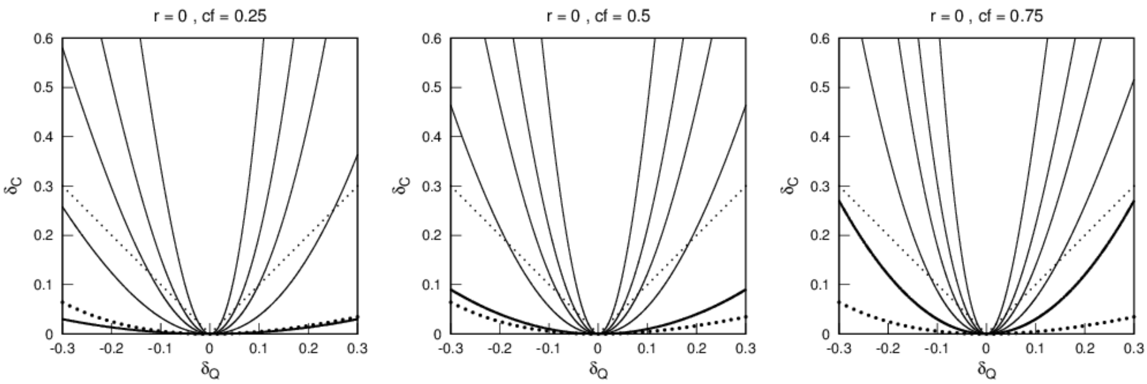

Figure 3: $\delta_C(\delta_Q)$ for normal distribution when $r = 0$

Figure 3 demonstrates the special case of $r = 0$. The construction of the diagrams is very similar to that of Figure 2. The magnitude of cost deviation decreases, but still remains at a much higher level than the benchmarks. Our observations with Figure 2 (and associated conclusions) hold for this special case.

7 Order quantity deviation: Some remarks

Figure 2 demonstrates that the penalty for deviation from the optima in the newsboy model in a "common" setting (i.e., normal demand distribution and practical values for model parameters) can be very high. However, the knowledge of cost deviation without any understanding order quantity deviation is incomplete. In this section, we study order quantity deviation.

Let $Z = (X - \mu)/\sigma$ be the standardized random variable associated with X. $E[Z] = 0$ and $Var(Z) = 1$. Let F_z be the distribution function associated with Z. $F_z(z) = F(\mu + z\sigma)$. The optimum order quantity in newsboy problem can be expressed as $Q^* = \mu + \sigma F_z^{-1}(cf)$. The operational order quantity is given by $\widehat{Q}^* = \widehat{\mu} + \widehat{\sigma} F_z^{-1}(\widehat{cf})$. It is assumed that the form of demand distribution is correctly known. Using these expressions of $\widehat{Q}^*$ and Q^*, order quantity deviation, $\delta_Q = (\widehat{Q}^* - Q^*)/Q^*$ can be expressed as

$$\delta_Q = \frac{\{\mu(1+\delta_\mu) - \mu\} + \{\sigma(1+\delta_\sigma)F_z^{-1}(cf(1+\delta_{cf})) - \sigma F_z^{-1}(cf)\}}{\mu + \sigma F_z^{-1}(cf)}$$
$$\Rightarrow \delta_Q = \frac{\delta_\mu + c_v\{(1+\delta_\sigma)F_z^{-1}(cf(1+\delta_{cf})) - F_z^{-1}(cf)\}}{1 + c_v F_z^{-1}(cf)}. \tag{7}$$

Above equation can be rewritten as $\delta_Q = (\delta_\mu + c_v \delta_{\sigma\times cf})/\{1 + c_v F_z^{-1}(cf)\}$, where $\delta_{\sigma\times cf} = (1+\delta_\sigma)F_z^{-1}(cf(1+\delta_{cf})) - F_z^{-1}(cf)$ is the joint impact of δ_σ and δ_{cf} on δ_Q. Impact of δ_μ and $\delta_{\sigma\times cf}$ on δ_Q is straight forward; δ_Q increases in both. If $c_v < 1$, impact of δ_μ is stronger than that of $\delta_{\sigma\times cf}$ (assuming magnitudes of δ_μ and $\delta_{\sigma\times cf}$ of same level). Thus, mean demand may be the most influential parameter in the newsboy model if c_v is not very high.

δ_σ and δ_{cf} impact $\delta_{\sigma\times cf}$ in a complex manner; a thorough investigation would require a dedicated study. Here, we demonstrate the influence of $\delta_\sigma, \delta_{cf}$ on $\delta_{\sigma\times cf}$ and $\delta_\mu, \delta_{\sigma\times cf}$ on δ_Q with an example. Before that, it should be noted that δ_{cf} is determined by interaction of δ_{c_u} and δ_{c_o}. $\delta_{cf} = \widehat{cf}/cf - 1$ can be expressed as

$$\delta_{cf} = \frac{\widehat{c_u}(c_o + c_u)}{(\widehat{c_o} + \widehat{c_u})c_u} - 1 = \frac{\widehat{c_u}c_o - c_u\widehat{c_o}}{\{(1+\delta_{c_o}) + (c_u/c_o)(1+\delta_{c_u})\}c_u c_o} = \frac{(1+\delta_{c_u}) - (1+\delta_{c_o})}{1 + \delta_{c_o} + \frac{cf}{1-cf}(1+\delta_{c_u})}$$
$$\Rightarrow \delta_{cf} = \frac{(1-cf)(\delta_{c_u} - \delta_{c_o})}{1 + cf\delta_{c_u} + (1-cf)\delta_{c_o}}. \tag{8}$$

Two properties can be readily observed about δ_{cf}: i) high cf leads to lower magnitude level of δ_{cf} compared to low cf and ii) same signs of δ_{c_u} and δ_{c_o} leads to lower magnitude level of δ_{cf} compared to opposite signs. Our demonstration confirms these observations.

Table 2 demonstrates δ_Q in different deviation scenarios for normal demand distribution with $c_v = 0.25$ (no truncation). $|\delta_\mu| = |\delta_\sigma| = |\delta_{c_u}| = |\delta_{c_o}| = 10\%$ taken. $\Delta = c_v\delta_{\sigma\times cf}$ represents the joint impact of δ_σ and δ_{cf}. Table 2 verifies our observations regarding δ_{cf}. Mean demand emerges as the most important parameter as $|\delta_\mu| > |\Delta|$ in every scenario; also, the signs of δ_μ and δ_Q are always same. It can also be noticed that $|\delta_Q|$ is higher when δ_μ and δ_{cf} are of same sign (compared to the case of opposing signs). This happens when i) $\delta_\mu < 0, \delta_{c_u} < 0, \delta_{c_o} > 0$ or ii) $\delta_\mu > 0, \delta_{c_u} > 0, \delta_{c_o} < 0$ (highlighted rows in Table 2).

Table 2: δ_Q for normal demand distribution

Deviation scenario				Low cf (0.25)			Medium cf (0.5)			High cf (0.75)		
δ_μ	δ_σ	δ_{c_u}	δ_{c_o}	δ_{cf}	Δ	δ_Q	δ_{cf}	Δ	δ_Q	δ_{cf}	Δ	δ_Q
−10	−10	−10	−10	0	1.7	−10	0	0	−10	0	−1.7	−10
−10	−10	−10	10	−14	−0.9	−13	−10	−2.8	−13	−5.3	−4.4	−12
−10	−10	10	−10	16	4.4	−6.8	10	2.8	−7.2	4.8	0.9	−7.7
−10	−10	10	10	0	1.7	−10	0	0	−10	0	−1.7	−10
−10	10	−10	−10	0	−1.7	−14	0	0	−10	0	1.7	−7.1
−10	10	−10	10	−14	−4.9	−18	−10	−3.5	−13	−5.3	−1.6	−9.9
−10	10	10	−10	16	1.6	−10	10	3.5	−6.5	4.8	4.9	−4.4
−10	10	10	10	0	−1.7	−14	0	0	−10	0	1.7	−7.1
10	−10	−10	−10	0	1.7	14	0	0	10	0	−1.7	7.1
10	−10	−10	10	−14	−0.9	11	−10	−2.8	7.2	−5.3	−4.4	4.8
10	−10	10	−10	16	4.4	17	10	2.8	13	4.8	0.9	9.4
10	−10	10	10	0	1.7	14	0	0	10	0	−1.7	7.1
10	10	−10	−10	0	−1.7	10	0	0	10	0	1.7	10
10	10	−10	10	−14	−4.9	6.1	−10	−3.5	6.5	−5.3	−1.6	7.2
10	10	10	−10	16	1.6	14	10	3.5	13	4.8	4.9	13
10	10	10	10	0	−1.7	10	0	0	10	0	1.7	10

Magnitude of δ_Q in Table 2 reconfirms the necessity of sensitivity analysis of the newsboy model. If magnitude level of order quantity deviation were extremely low, study of cost deviation would have been a mere theoretical exercise.

8 Discussion

Existing literature on sensitivity analysis of the newsboy model is limited. We have mere understanding of direction of cost and order quantity deviations as parameter estimates deviate from their true values. This work contributes to the literature in three distinct ways; we list them below and discuss their implications.

First, necessary and sufficient conditions for symmetry (skewness) of cost deviation are identified (proposition 1 and 2). According to these conditions, for symmetric unimodal demand distributions (e.g., normal distribution), it is better to under-estimate the order quantity (than over-estimating) if $cf < 1/2$ and it is better to over-estimate the order quantity (than under-estimating) if $cf > 1/2$ (corollary 1). Based on experiments, Schweitzer & Cachon (2000) reported that managers order more than the optimal when $cf < 1/2$ and less than the optimal when $cf > 1/2$. They considered symmetric unimodal demand distribution. Kevork (2010) developed estimators for order quantity and expected profit for normal demand distribution. He, too, observed that the order quantity is over-estimated when $cf < 1/2$ and under-estimated when $cf > 1/2$. Our finding suggests that expected profit maximizing managers are better off doing the opposite if one is uncertain about the optimal order quantity.

Second, we demonstrate magnitude level of cost deviation for normal demand distribution (Figure 2) and compared it with two benchmarks (cost deviation of the EOQ model and ± 1 slope lines). The EOQ model is robust, i.e., increase in inventory costs due to sub-optimal ordering decision is very low. Popular stochastic inventory models like (r,Q) and (s,S) inventory systems have been found to be more robust than the EOQ model (by Zheng, 1992; F. Chen & Zheng, 1997). Newsboy model is an exception; it is much more sensitive to sub-optimal ordering decisions than the EOQ model. In fact, in many situations, magnitude of cost deviation exceeds the magnitude of the input error. This is a clear signal to the practitioners to take proper care of the parameter estimation processes so that the order quantity deviation is small. Cost deviation increases with ratio of demand limits (r) and decreases with coefficient of variation (c_v). This further signals the managers that the newsboy model is least robust in high r - low c_v scenarios. However, in absence of thorough investigation into influence of these factors on order quantity deviation, we can not be conclusive about their impact.

Unlike the results regarding symmetry (skewness) of cost deviation, the results regarding magnitude of cost deviation are applicable only for normal demand distribution. However, due to versatility of shape of normal distribution (Figure 1), we can expect these conclusions to hold for any symmetric unimodal distribution.

Third, a brief study of order quantity deviation is carried out. It has been observed that the order quantity deviation can be high for moderate error in parameter estimation (Table 2). This demonstration along with the results regarding cost deviation clearly suggests that the penalty for error in parameter estimation can be very high in the newsboy model; input error is amplified multiple times and a very high cost deviation is observed. Based on Table 2, we find two scenarios to be most undesirable: i) $\delta_\mu < 0, \delta_{c_u} < 0, \delta_{c_o} > 0$ and ii) $\delta_\mu > 0, \delta_{c_u} > 0, \delta_{c_o} < 0$; these situations should be avoided. We also found that mean demand is the most influential parameter in the newsboy model; keeping δ_μ at a low magnitude level helps limiting order quantity deviation, thereby limiting cost deviation.

9 Conclusion

We perform sensitivity analysis of the newsboy model, one of the most popular inventory models in the literature. We convincingly demonstrate that the newsboy model is sensitive to sub-optimal ordering decisions, much more sensitive than the EOQ model. Necessary and sufficient conditions for symmetry (skewness) of cost deviation are identified and magnitude level of cost deviation is elaborately demonstrated for normal demand distribution. It has also been manifested that order quantity deviation can be high for moderate input error.

Our conclusions regarding magnitude of cost deviation are limited to symmetric unimodal demand distributions. However, most demand distributions are unimodal and symmetric; hence, our study addresses the prevalent case. We did not study the distribution of cost deviation in this work; it can be considered as a future extension.

We focused on the impact of sub-optimal ordering decisions on expected mismatch cost. The investigation into the impact of parameter estimation error on order quantity lacks depth; however, it is sufficient to establish relevance of the study of cost deviation. Some interesting observations about order quantity deviation has been made. A thorough investigation will further enhance our understanding of sensitivity of the newsboy model.

This work establishes high sensitivity of the newsboy model to sub-optimal ordering decisions. To improve decision making, parameter estimation error needs to be reduced. Since the newsboy model is multi-parameter model, the question of parameter importance becomes relevant. Another way to tackle this issue is to minimize the cost deviation instead minimizing the cost. Lowe, Schwarz, & McGavin (1988) took this approach assuming uncertain cost parameters; their work can be extended to the case of uncertain demand parameters.

Appendix A

Profit in the newsboy model is maximum possible profit (when $Q = X$, i.e., no mismatch between demand and supply) less mismatch cost. Let the unit profit be m and profit for order quantity Q be $\Pi(Q)$. Then $\Pi(Q) = mX - C(Q) \Rightarrow E[\Pi(Q)] = m\mu - E[C(Q)]$. $E[\Pi(Q)]$ is concave as $E[C(Q)]$ is convex. The cost minimizing order quantity maximizes profit, i.e., $F(Q^*) = cf$ and $E[\Pi(Q^*)] = m\mu - E[C(Q^*)]$. Now,

$$\delta_\Pi = \frac{E[\Pi(Q)] - E[\Pi(Q^*)]}{E[\Pi(Q^*)]} = \frac{E[C(Q^*)]}{E[\Pi(Q^*)]}\frac{E[C(Q^*)] - E[C(Q)]}{E[C(Q^*)]} = -\frac{E[C(Q^*)]}{E[\Pi(Q^*)]}\delta_C.$$

$E[C(Q^*)]/E[\Pi(Q^*)]$ is constant. Using above relation, sensitivity analysis results of δ_C can be easily converted into corresponding results of δ_Π.

Appendix B

First, we show strict convexity of $E[C(Q)]$ in $[a,b]$. Let us consider arbitrary $Q_1, Q_2 \in [a,b]$ $(Q_1 \neq Q_2)$ and $\lambda \in (0,1)$. Let $Q = \lambda Q_1 + (1-\lambda)Q_2$. Without loss of generality, let us assume that $Q_1 < Q_2$. Then $Q_1 < Q < Q_2$. Using (2),

$$\begin{aligned}
&\lambda E[C(Q_1)] + (1-\lambda)E[C(Q_2)] = c_u[\lambda(\mu - Q_1) + (1-\lambda)(\mu - Q_2)] \\
&\quad + (c_o + c_u)\left\{\lambda \int_a^{Q_1} (Q_1 - x)f(x)\mathrm{d}x + (1-\lambda)\int_a^{Q_2} (Q_2 - x)f(x)\mathrm{d}x\right\}. \\
&= c_u(\mu - Q) + (c_o + c_u)\left[\left\{\lambda \int_a^{Q} (Q_1 - x)f(x)\mathrm{d}x + (1-\lambda)\int_a^{Q} (Q_2 - x)f(x)\mathrm{d}x\right\}\right. \\
&\quad \left. + \left\{\lambda \int_{Q_1}^{Q} (x - Q_1)f(x)\mathrm{d}x + (1-\lambda)\int_{Q}^{Q_2} (Q_2 - x)f(x)\mathrm{d}x\right\}\right] \\
&> c_u(\mu - Q) + (c_o + c_u)\int_a^{Q} (Q - x)f(x)\mathrm{d}x = E[C(Q)].
\end{aligned}$$

Since Q_1, Q_2, λ are arbitrary, $E[C(\lambda Q_1 + (1-\lambda)Q_2)] < \lambda E[C(Q_1)] + (1-\lambda)E[C(Q_2)]$ $\forall Q_1, Q_2 \in [a,b]$ $(Q_1 \neq Q_2)$ and $\forall \lambda \in (0,1)$. Hence, $E[C(Q)]$ is strictly convex in $[a,b]$.

Now, we establish optimality of $Q^* = F^{-1}(cf)$ in minimizing $E[C(Q)]$. Let $Q \in [a,b] \setminus \{Q^*\}$. Note that $F(x) < cf$ if $x < Q^*$ and $F(x) > cf$ if $x > Q^*$. Using (2) and (3),

$$\begin{aligned}
E[C(Q)] - E[C(Q^*)] &= -c_u Q + (c_o + c_u)\left\{\int_a^{Q} Qf(x)\mathrm{d}x - \int_{Q^*}^{Q} xf(x)\mathrm{d}x\right\} \\
&= (c_o + c_u)\left[-Qcf + QF(Q) - \left\{QF(Q) - Q^*cf - \int_{Q^*}^{Q} F(x)\mathrm{d}x\right\}\right] \\
&= (c_o + c_u)\int_{Q^*}^{Q} \{F(x) - cf\}\mathrm{d}x > 0.
\end{aligned}$$

So, $E[C(Q^*)] < E[C(Q)]$ $\forall Q \in [a,b] \setminus \{Q^*\}$. Since $E[C(Q)]$ minimizing order quantity is in $[a,b]$, $Q^* = F^{-1}(cf)$ minimizes $E[C(Q)]$.

If we relax the assumption that $f(x) > 0$ for almost all $x \in (a,b)$, $E[C(Q)]$ is convex in $[a,b]$ (not strictly convex) and $F(Q^*) = cf$ minimizes $E[C(Q)]$ (not $Q^* = F^{-1}(cf)$).

Appendix C

We prove lemma 1 for an arbitrary $F \in \mathcal{D}_{a,b}$.

If $x \in (a,\mu)$, $F(x) = \int_a^x f(y)\mathrm{d}y > 0 = F_0(x)$ as $f(y) > 0$ $\forall y \in (a,x]$. Due to convexity of F in $[a,\mu]$, $F(\lambda a + (1-\lambda)\mu) \leq \lambda F(a) + (1-\lambda)F(\mu) = (1-\lambda)/2$ $\forall \lambda \in (0,1)$. Denoting $\lambda a + (1-\lambda)\mu = x$, i.e., $\lambda = (\mu - x)/(\mu - a) = 1 - 2F_U(x)$, we get $F(x) \leq F_U(x)$ $\forall x \in (a,\mu)$. Hence, $F_0(x) < F(x) \leq F_U(x)$ if $x \in (a,\mu)$.

Similarly, due to concavity of F in $[\mu, b]$, $F(\lambda\mu + (1-\lambda)b) \geq \lambda F(\mu) + (1-\lambda)F(b) = 1-\lambda/2\ \forall\lambda \in (0,1)$. Denoting $\lambda\mu + (1-\lambda)b = x$, i.e., $\lambda = (b-x)/(b-\mu) = 2\{1-F_U(x)\}$, we get $F(x) \geq F_U(x)\ \forall x \in (\mu, b)$. If $x \in (\mu, b)$, $F(x) = 1-\int_x^b f(y)\mathrm{d}y < 1 = F_0(x)$ as $f(y) > 0\ \forall y \in [x, b)$. Hence, $F_U(x) \leq F(x) < F_0(x)$ if $x \in (\mu, b)$.

Appendix D

We say that $F_{N(c_v)} \to F$ if $F_{N(c_v)}(x) \to F(x)\ \forall x \in [a, b]$.

Since f exists, for the first part of lemma 2, it is sufficient to show that $f_{N(c_v)}(x) \to f_U(x)$ as $c_v \to \infty$ for an arbitrary $x \in [a, b]$. Let μ_0 and σ_0 be the mean and standard deviation of the underlying normal distribution. $\mu_0 = \mu = (a+b)/2$ and $\sigma_0 = c_v\mu_0$. Let $\zeta_x = (x-\mu_0)/\sigma_0$.

$$\lim_{c_v\to\infty} f_{N(c_v)}(x) = \lim_{c_v\to\infty} \frac{1}{\sigma_0}\frac{\phi(\zeta_x)}{\Phi(\zeta_b)-\Phi(\zeta_a)} = \lim_{c_v\to\infty} \frac{2}{c_v(a+b)}\frac{\phi\left(\frac{x-\mu_0}{c_v(a+b)}\right)}{\Phi\left(\frac{b-a}{c_v(a+b)}\right)-\Phi\left(-\frac{b-a}{c_v(a+b)}\right)}$$
$$= \frac{2}{a+b}\lim_{c_v\to\infty}\exp\left(-\frac{(x-\mu_0)^2}{2c_v^2(a+b)^2}\right)\lim_{c_v\to\infty}\frac{\frac{1}{c_v}}{\Phi\left(\frac{k}{c_v}\right)-\Phi\left(-\frac{k}{c_v}\right)},\ \text{where } k = \frac{b-a}{a+b}$$
$$= \frac{2}{a+b}\lim_{c_v\to\infty}\frac{\frac{1}{c_v}}{\int_{-k/c_v}^{k/c_v}\exp\left(-\frac{x^2}{2}\right)\mathrm{d}x} = \frac{2}{a+b}\lim_{c_v\to\infty}\frac{-\frac{1}{c_v^2}}{-\frac{2k}{c_v^2}\exp\left(-\frac{k^2}{2c_v^2}\right)}\ \text{(by l'Hospital's rule)}$$
$$= \frac{1}{(a+b)k}\lim_{c_v\to\infty}\exp\left(-\frac{k^2}{2c_v^2}\right) = \frac{1}{b-a} = f_U(x).$$

For the second part of lemma 2, we need to show that $F_{N(c_v)}(x) \to F_0(x)\ \forall x \in [a, b] \setminus (\mu-\epsilon, \mu+\epsilon)$ for any small (but fixed) $\epsilon > 0$ as $c_v \to 0^+$.

$$\lim_{c_v\to 0^+} F_{N(c_v)}(x) = \lim_{c_v\to 0^+}\frac{\Phi(\zeta_x)-\Phi(\zeta_a)}{\Phi(\zeta_b)-\Phi(\zeta_a)} = \lim_{C_v\to 0^+}\frac{\Phi(\zeta_x)-\Phi(-k/c_v)}{\Phi(k/c_v)-\Phi(-k/c_v)},\ \text{where } k = \frac{b-a}{a+b}$$
$$= \frac{\lim_{c_v\to 0^+}\Phi(\zeta_x)-0}{1-0} = \lim_{c_v\to 0^+}\Phi\left(\frac{x-\mu}{c_v(a+b)}\right).$$

When $x \in [a, \mu-\epsilon]$, $-(b-a)/2 \leq x-\mu \leq -\epsilon$ and when $x \in [\mu+\epsilon, b]$, $\epsilon \leq x-\mu \leq (b-a)/2$. Then $\epsilon \leq |x-\mu| \leq (b-a)/2$. So $|x-\mu|/(a+b)$ is positive and bounded. As Φ is continuous,

$$\text{If } x \in [a, \mu-\epsilon],\ \lim_{c_v\to 0^+} F_{N(c_v)}(x) = \Phi\left(\lim_{c_v\to 0^+}\frac{-|x-\mu|}{c_v(a+b)}\right) = 0 = F_0(x).$$
$$\text{If } x \in [\mu+\epsilon, b],\ \lim_{c_v\to 0^+} F_{N(c_v)}(x) = \Phi\left(\lim_{c_v\to 0^+}\frac{|x-\mu|}{c_v(a+b)}\right) = 1 = F_0(x).$$

One interesting observation: $\lim_{c_v\to 0^+} F_{N(c_v)}(\mu) = \Phi(\lim_{c_v\to 0^+} 0/c_v) = \Phi(\lim_{c_v\to 0^+} 0/1) = 1/2$ (by l'Hospital's rule). For low values of c_v, $F_{N(c_v)}$ rapidly increases from 0^+ to 1^- around the mean, maintaining $F_{N(c_v)}(\mu) = 1/2$. F_0, on the other hand, jumps from 0 to 1 at $x = \mu$.

This mismatch can be reduced to any level (but can not be eliminated) by reducing c_v.

Appendix E

We derive (5) using the definition of cost deviation itself. Let $\zeta_x = (x - \mu_0)/\sigma_0$ and $Z = \Phi(\zeta_b) - \Phi(\zeta_a)$. Then $f(x) = \phi(\zeta_x)/(\sigma_0 Z)$ and $F(x) = \{\Phi(\zeta_x) - \Phi(\zeta_a)\}/Z$. First, we evaluate $\int_a^Q (Q - x) f(x) \mathrm{d}x$ for normal distribution.

$$\begin{aligned}
&\int_a^Q (Q - x) f(x) \mathrm{d}x = (Q - \mu_0) F(Q) + \int_a^Q (\mu_0 - x) \frac{\phi(\zeta_x)}{\sigma_0 Z} \mathrm{d}x \\
&= (Q - \mu_0) F(Q) - \frac{\sigma_0}{\sqrt{2\pi} Z} \int_a^Q \frac{x - \mu_0}{\sigma_0^2} \exp\left(-\frac{1}{2} \left(\frac{x - \mu_0}{\sigma_0} \right)^2 \right) \mathrm{d}x \\
&= (Q - \mu_0) F(Q) - \frac{\sigma_0}{\sqrt{2\pi} Z} \int_{\frac{1}{2}\left(\frac{a-\mu_0}{\sigma_0}\right)^2}^{\frac{1}{2}\left(\frac{Q-\mu_0}{\sigma_0}\right)^2} \exp(-y) \mathrm{d}y \quad \text{replacing } \frac{1}{2}\left(\frac{x - \mu_0}{\sigma_0}\right)^2 = y \\
&= (Q - \mu_0) F(Q) - \frac{\sigma_0}{Z} \frac{1}{\sqrt{2\pi}} \left\{ \exp\left(-\frac{1}{2} \left(\frac{a - \mu_0}{\sigma_0} \right)^2 \right) - \exp\left(-\frac{1}{2} \left(\frac{Q - \mu_0}{\sigma_0}^2 \right) \right) \right\} \\
&= (Q - \mu_0) F(Q) + \frac{\sigma_0}{Z} \{\phi(\zeta_Q) - \phi(\zeta_a)\}.
\end{aligned}$$

Using (2) and above expression,

$$\begin{aligned}
E[C(Q)] &= (c_o + c_u) \left[cf(\mu - Q) + \int_a^Q (Q - x) f(x) \mathrm{d}x \right] \\
&= (c_o + c_u) \left[(Q - \mu_0)\{F(Q) - cf\} + \frac{\sigma_0}{Z}\{\phi(\zeta_Q) - \phi(\zeta_a)\} \right] \quad \text{as } \mu = \mu_0.
\end{aligned}$$

Putting $F(Q^*) = cf$ in the above expression,

$$E[C(Q^*)] = (c_o + c_u) \frac{\sigma_0}{Z} \{\phi(\zeta_{Q^*}) - \phi(\zeta_a)\}.$$

Since $\delta_C = (E[C(Q)] - E[C(Q^*)])/(E[C(Q^*)])$,

$$\begin{aligned}
\delta_C(\delta_Q) &= \frac{(Q - \mu_0)\{F(Q) - cf\} + (\sigma_0/Z)\{\phi(\zeta_Q) - \phi(\zeta_{Q^*})\}}{(\sigma_0/Z)\{\phi(\zeta_{Q^*}) - \phi(\zeta_a)\}} \\
&= \frac{\zeta_Q\{\Phi(\zeta_Q) - \Phi(\zeta_{Q^*})\} + \{\phi(\zeta_Q) - \phi(\zeta_{Q^*})\}}{\phi(\zeta_{Q^*}) - \phi(\zeta_a)}, \quad \text{where } Q = Q^*(1 + \delta_Q).
\end{aligned}$$

Now, $cf = F(Q^*) = (1/Z)\{\Phi(\zeta_{Q^*}) - \Phi(\zeta_a)\} \Rightarrow \zeta_{Q^*} = \Phi^{-1}\left((1 - cf)\Phi(\zeta_a) + cf\Phi(\zeta_b)\right)$. Then $\zeta_Q = (Q - \mu_0)/\sigma_0 = (Q^* - \mu_0)/\sigma_0 + (Q^*/\sigma_0)\delta_Q = \zeta_{Q^*} + (\zeta_{Q^*} + \mu_0/\sigma_0)\delta_Q = \zeta_{Q^*}(1 + \delta_Q) + \delta_Q/c_v$. Since $\mu_0 = (a + b)/2$, $\zeta_a = -k/c_v$ and $\zeta_b = k/c_v$, where $k = (1 - r)/(1 + r)$. Then

$$\delta_C(\delta_Q) = \frac{z\{\Phi(z) - \Phi(z^*)\} + \{\phi(z) - \phi(z^*)\}}{\phi(z^*) - \phi(-k/c_v)},$$

where $z^* = \Phi^{-1}\left((1-cf)\Phi(-k/c_v) + cf\Phi(k/c_v)\right)$ and $z = z^*(1+\delta_Q) + \delta_Q/c_v$.

Appendix F

For uniformly distributed demand, $F(x) = (x-a)/(b-a)$. Using (4),

$$\begin{aligned}
\delta_C(\delta_Q) &= \frac{\int_{Q^*}^{Q}\{(x-a)/(b-a) - cf\}\mathrm{d}x}{\{(a+b)/2 - Q^*\}cf + \int_a^{Q^*}(x-a)/(b-a)\mathrm{d}x}, \quad \text{where } Q = Q^*(1+\delta_Q) \\
&= \frac{\int_{Q^*}^{Q}(x-Q^*)/(b-a)\mathrm{d}x}{(1/2-cf)(b-a)cf + \int_a^{Q^*}(x-a)/(b-a)\mathrm{d}x} \quad \text{as } cf = (Q^*-a)/(b-a) \\
&= \frac{(Q-Q^*)^2/2}{cf(1/2-cf)(b-a)^2 + (Q^*-a)^2/2} = \frac{\{a+cf(b-a)\}^2}{cf(1-cf)(b-a)^2}\delta_Q^2.
\end{aligned}$$

Dividing the numerator and the denominator by b, we get (6).

References

Borgonovo, E., & Peccati, L. (2007). Global sensitivity analysis in inventory management. *International Journal of Production Economics*, *108*(1-2), 302–313.

Bozarth, C. C., & Handfield, R. B. (2006). *Introduction to Operations and Supply Chain Management* (1st ed.). Prentice Hall.

Chen, F., & Zheng, Y.-S. (1997). Sensitivity analysis of an (s,S) inventory model. *Operations Research Letters*, *21*(1), 19–23.

Chen, L.-H., & Chen, Y.-C. (2010). A multiple-item budget-constraint newsboy problem with a reservation policy. *Omega*, *38*(6), 431–439.

Choi, T.-M. (Ed.). (2012). *Handbook of Newsvendor Problems* (1st ed.). Springer.

Dobson, G. (1988). Sensitivity of the EOQ model to parameter estimates. *Operations Research*, *36*(4), 570–574.

Eeckhoudt, L., Gollier, C., & Schlesinger, H. (1995). The risk-averse (and prudent) newsboy. *Management Science*, *41*(5), 786–794.

Gerchak, Y., & Mossman, D. (1992). On the effect of demand randomness on inventories and costs. *Operations Research*, *40*(4), 804–807.

Gkedenko, B. V., & Kolmogorov, A. N. (1954). *Limit Distributions for Sums of Independent Random Variables (English translation by K. L. Chung)*. Massachusetts: Addison-Wesley Publishing Company.

Kevork, I. S. (2010). Estimating the optimal order quantity and the maximum expected profit for single-period inventory decisions. *Omega*, *38*(3-4), 218–227.

Khouja, M. (1999). The single-period (news-vendor) problem: Literature review and suggestions for future research. *Omega*, *27*(5), 537–553.

Lau, A. H.-L., & Lau, H.-S. (2002). The effects of reducing demand uncertainty in a manufacturer-retailer channel for single-period products. *Computers & Operations Research*, *29*(11), 1583–1602.

Lowe, T. J., & Schwarz, L. B. (1983). Parameter estimation for the EOQ lot-size model: Minimax and expected value choices. *Naval Research Logistics Quarterly*, *30*(2), 367–376.

Lowe, T. J., Schwarz, L. B., & McGavin, E. J. (1988). The determination of optimal base-stock inventory policy when the costs of under- and oversupply are uncertain. *Naval Research Logistics*, *35*(4), 539–554.

Nahmias, S. (2001). *Production and Operations Analysis* (4th ed.). Boston: McGraw-Hill.

Porteus, E. L. (1990). Stochastic inventory theory. In D. P. Heyman & M. J. Sobel (Eds.), *Handbooks in or & ms, vol. 2* (pp. 610–628). Amsterdam: North-Holland Pub. Co.

Prichard, J. W., & Eagle, R. H. (1965). *Modern Inventory Management* (1st ed.). New York: John Wiley & Sons.

Qin, Y., Wang, R., Vakharia, A. J., Chen, Y., & Seref, M. M. H. (2011). The newsvendor problem: Review and directions for future research. *European Journal of Operational Research*, *213*(2), 361–374.

Ridder, A., van der Laan, E., & Salomon, M. (1998). How larger demand variability may lead to lower costs in the newsvendor problem. *Operations Research*, *46*(6), 934–936.

Schweitzer, M. E., & Cachon, G. P. (2000). Decision bias in the newsvendor problem with a known demand distribution: Experimental evidence. *Management Science*, *46*(3), 404–420.

Silver, E. A., Pyke, D. F., & Peterson, R. (1998). *Inventory Management and Production Planning and Scheduling* (3rd ed.). New York: John Wiley & Sons.

Zheng, Y.-S. (1992). On properties of stochastic inventory systems. *Management Science*, *38*(1), 87–103.