A Quadratic Programming Formulation of the Equidistant Bi-directional Loop Layout Problem

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Abstract

A loop layout is a common layout used in flexible manufacturing. In such a layout, a set of stations or facilities are to be arranged in a closed loop so that the total cost of flow between each pair of facilities is minimized. The most common mathematical programming formulation of the problem is based on a quadratic assignment formulation. In this paper, we modify that formulation taking advantage of the structure of the problem.

Keywords: Flexible manufacturing, facility layout, bi-directional closed loop layout.

1 Introduction

The cost of material handling is one of the main contributors to the cost of operations in flexible manufacturing, often varying between 15 to 70% of the total cost (see, e.g., Kim and Kim, 2000). This cost depends on the efficiency with which a set of facilities (machines, stations, etc.) are assigned to candidate positions, so that a material handling agent can transfer work in progress from one facility to another. The candidate positions are arranged in a variety of ways, leading to different layout designs.

The most common is the loop layout (see Figure 1), in which the candidate positions are arranged around a closed loop. This layout is often seen when there is a radial arm which handles the inter-station flow. The radial arm can either rotate in one direction, leading to a uni-directional loop layout, or in both directions, leading to a bi-directional loop layout. The next most common layout
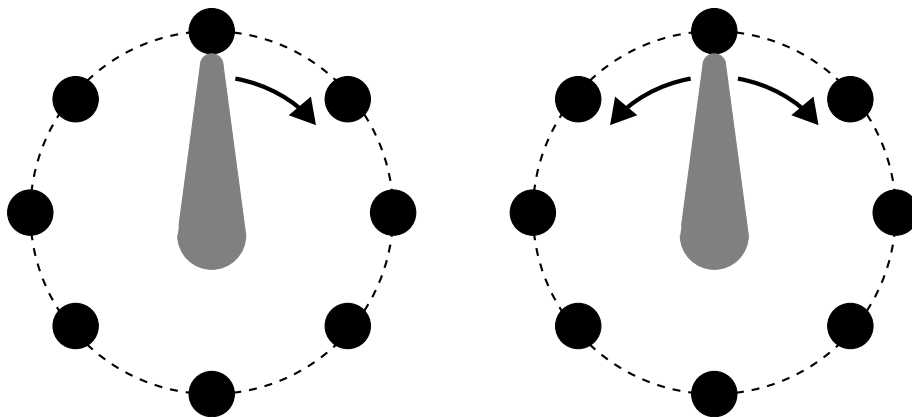

Figure 1: Loop layouts: Unidirectional (left) and Bi-directional (right)

is a linear layout (see Figure 2), in which positions are arranged in a line. This is common when the inter-station flow is being handled on a straight conveyor belt. There are two variants of this layout,
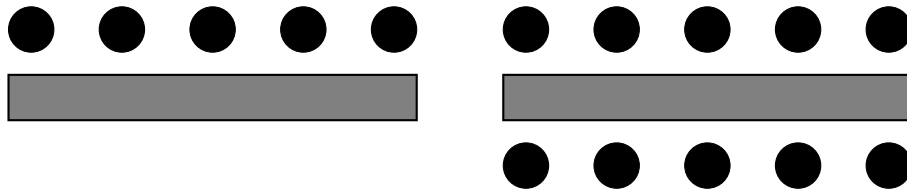
Figure 2: Linear layouts: Single row (left) and Double row (right)

the single row facility layout, in which the positions are on one side of the belt, and the double row layout, in which the positions are on both sides of the belt. A third layout is called the semi-circular layout (see Figure 3). This is a variant of the loop layout, where the positions are arranged in a semi-circle rather than a complete loop.
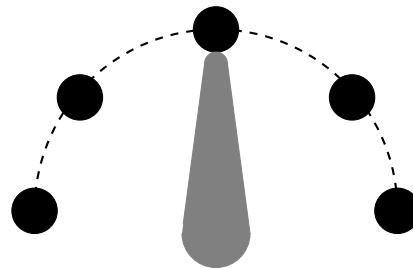

Figure 3: Semi-circular layout

The loop layout problem is one of assigning facilities to positions such that the total cost of material transfer among facilities is minimized. The total cost of flow between a pair of facilities is taken as the product of the amount of flow between them and the distance between the facilities, given their positions. Formally stated the problem is the following:

The Loop Layout Problem

Given   A set of N facilities, labeled 1 through N, N positions on the circumference of a circle, and the flow f_{ij} between facilities i and j for all facility pairs $\{i, j\}$.

Required An assignment of facilities to positions such that the total cost of sending flow among all pairs of facilities is minimized.

The loop layout problem is called uni-directional or bi-directional depending on how the distance between two facilities is computed. Suppose the facilities are at positions k and l. If the distance between the positions is computed by traversing the circumference of the cycle only in the clockwise direction (or, only in the counter-clockwise direction), the problem is called the uni-directional loop layout problem. However, if the distance between the two positions is the smaller among two distances, one when traveling clockwise from k to l, and the other when traveling counter-clockwise from k to l, then the problem is called a bi-directional loop layout problem.

Several mathematical programming formulations of layout problems have been presented in the literature, although the number of formulations of the linear layout problem (both single row an double row layouts) is more than that for loop layout problems, especially in recent times. We refer an interested reader to Anjos et al. (2005), Amaral (2008), Amaral (2009), Amaral and Letchford (2012), Hungerländer and Rendl (2011), etc.

Afentakis (1989) first proposed a discrete formulation for the loop layout problem. Subsequently, Kaku and Rachamadugu (1992), Kiran et al. (1992), Kiran and Karabati (1993),Bozer and Rim

(1996), Altinel and Öncan (2005), and Öncan and Altinel (2008) have mostly modeled the unidirectional version problem as a quadratic assignment problem. Sharp and Liu (1990) and Kiran and Karabati (1993) have proposed alternative mixed integer programming models for the problem. Saravanan and Ganesh Kumar (2013) provides an up to date review of the literature on loop layout problems.

In this paper, we provide a formulation of a special case of the bi-directional loop layout problem called the equidistant bi-directional loop layout problem (EBLLP). In this problem, each position is at equal distances from its neighboring positions. Our formulation makes use of the various symmetry conditions in the EBLLP. In Section 3, we provide an example explaining our formulation. The last section is a summary of the contribution of the paper.

2   A Mathematical Model for the EBLLP

Let the indices l and k denote positions from 1 through N, and indices i and j denote facilities from 1 through N occupying positions. x_{pq}, $1 \leq p, q \leq N$ are decision variables that obtain a value of 1 if facility q occupies position p and 0 otherwise. The term d_{kl} denotes the distance between positions k and l, $(l > k)$ and is calculated as

$$d_{kl} = \frac{2\pi r}{N} \times \min\{(l - k), (N + k - l)\}$$

where r is the radius of the circle on which the facilities are arranged.

A mathematical programming formulation of the circular ordering problem is given below.

$$\text{MODEL1: Minimize } Z = \sum_{k=1}^{N-1} \sum_{l=k+1}^{N} \sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} d_{kl} f_{ij} x_{ki} x_{lj}$$

Subject to

$$\sum_{i=1}^{N} x_{ki} = 1 \quad \text{for } 1 \leq k \leq N \tag{1}$$

$$\sum_{k=1}^{N} x_{ki} = 1 \quad \text{for } 1 \leq i \leq N \tag{2}$$

$$x_{ki} \in \{0, 1\} \quad \text{for } 1 \leq k, i \leq N.$$

Here z is the total "communication" cost incurred by the solution. Constraint set (2) ensures that each position has one facility assigned to it, while constraint set (1) ensures that each facility occupies only one position in the layout.

In the circular ordering problem, $f_{ij} = f_{ji}$ for $1 \leq i, j, \leq N$. So

$$\sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} f_{ij} x_{ki} x_{lj} = 2 \times \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} f_{ij} x_{ki} x_{lj}.$$

Also, without loss of generality, we can specify that facility 1 occupies position 1 in the layout. If this is specified, $x_{11} = 1$, so that $\sum_{k=2}^{N} x_{k1} = \sum_{i=2}^{N} x_{1i} = 0$. The model can then be re-written as

follows.

$$\text{MODEL2: Minimize } Z = \sum_{l=2}^{N} \sum_{j=2}^{N} d_{1l} f_{1j} x_{lj} + 2 \sum_{k=2}^{N-1} \sum_{l=k+1}^{N} \sum_{i=2}^{N-1} \sum_{j=i+1}^{N} d_{kl} f_{ij} x_{ki} x_{lj}$$

Subject to

$$\sum_{i=2}^{N} x_{ki} = 1 \quad \text{for } 2 \leq k \leq N \tag{3}$$

$$\sum_{k=2}^{N} x_{ki} = 1 \quad \text{for } 2 \leq i \leq N \tag{4}$$

$$x_{ki} \in \{0, 1\} \quad \text{for } 2 \leq k, i \leq N.$$

Constraint set (3) can next be used to modify the model and reduce the number of variables. For each k, $2 \leq k \leq N$,

$$x_{kN} = 1 - \sum_{i=2}^{N-1} x_{ki}. \tag{5}$$

So, this removes $N - 1$ variables from the formulation. However, the constraints that $x_{kN} \in \{0, 1\}$ gets modified to constraints of the form

$$\sum_{i=2}^{N-1} x_{ki} \leq 1 \text{ for each } k = 2, \ldots, N. \tag{6}$$

In a similar fashion, from constraint set (4) we see that for each i, $2 \leq i \leq N$,

$$x_{Ni} = 1 - \sum_{k=2}^{N-1} x_{ki}, \tag{7}$$

and the restriction on x_{Ni} to be binary yields constraints of the form

$$\sum_{k=2}^{N-1} x_{ki} \leq 1 \text{ for each } i = 2, \ldots, N. \tag{8}$$

Also, since at most one among x_{52} through $x_{5(n-1)}$ can acquire a value of 1 in any feasible solution, we have

$$\sum_{k=2}^{N-1} \sum_{i=2}^{N-1} x_{k1} \geq N - 3. \tag{9}$$

The objective function in MODEL2 can be written as $Z = Z_1 + 2Z_r$ where $Z_1 = \sum_{l=2}^{N} \sum_{j=2}^{N} d_{1l} f_{1j} x_{lj}$ is the cost of communication between facility 1 (located at position 1) with the other facilities, and $2Z_r = 2 \sum_{k=2}^{N-1} \sum_{l=k+1}^{N} \sum_{i=2}^{N-1} \sum_{j=i+1}^{N} d_{kl} f_{ij} x_{ki} x_{lj}$ is the cost of communication between all facilities except facility 1.

Now $Z_1 = \displaystyle\sum_{l=2}^{N} \sum_{j=2}^{N} d_{1l} f_{1j} x_{lj}$

$= \displaystyle\sum_{l=2}^{N} d_{1l} \left\{ \sum_{j=2}^{N-1} f_{1j} x_{lj} + f_{1N} x_{lN} \right\}$

$= \displaystyle\sum_{l=2}^{N} d_{1l} \left\{ f_{1N} + \sum_{j=2}^{N-1} (f_{1j} - f_{1N}) x_{lj} \right\}$ (using eqn (5))

$= \displaystyle\sum_{l=2}^{N} d_{1l} f_{1N} + \sum_{l=2}^{N} d_{1l} \sum_{j=2}^{N-1} (f_{1j} - f_{1N}) x_{lj}$

$= \displaystyle\sum_{l=2}^{N} d_{1l} f_{1N} + \sum_{l=2}^{N-1} d_{1l} \sum_{j=2}^{N-1} (f_{1j} - f_{1N}) x_{lj} + d_{1N} \sum_{j=2}^{N-1} (f_{1j} - f_{1N}) x_{Nj}$

$= \displaystyle\sum_{l=2}^{N} d_{1l} f_{1N} + \sum_{l=2}^{N-1} d_{1l} \sum_{j=2}^{N-1} (f_{1j} - f_{1N}) x_{lj} +$

$\qquad d_{1N} \displaystyle\sum_{j=2}^{N-1} (f_{1j} - f_{1N}) \left(1 - \sum_{l=2}^{N-1} x_{lj} \right)$ (using eqn (7))

$= \displaystyle\sum_{l=2}^{N} d_{1l} f_{1N} + d_{1N} \sum_{j=2}^{N-1} (f_{1j} - f_{1N}) +$

$\qquad \displaystyle\sum_{l=2}^{N-1} d_{1l} \sum_{j=2}^{N-1} (f_{1j} - f_{1N}) x_{lj} - d_{1N} \sum_{j=2}^{N-1} (f_{1j} - f_{1N}) \sum_{l=2}^{N-1} x_{lj}$

$= \displaystyle\sum_{l=2}^{N} d_{1l} f_{1N} + d_{1N} \sum_{j=2}^{N-1} (f_{1j} - f_{1N}) + \sum_{j=2}^{N-1} \sum_{l=2}^{N-1} (d_{1l} - d_{1N})(f_{1j} - f_{1N}) x_{lj}.$ (10)

Note that $d_{1p} = d_{1(N+1-p)}$ for all i, $2 \leq i \leq N/2$. So the coefficients of x_{pj} and $x_{(N+1-p)j}$ in the expansion of $\sum_{l=2}^{N} \sum_{j=2}^{N} d_{1l} f_{1j} x_{lj}$ will be identical for all p, $2 \leq p \leq N/2$. So when equation (7) is applied to obtain expression (10) for Z_1, the expression will not include terms $x_{2,j}$, $j = 2, \ldots, N-1$.

Similarly $Z_r = \displaystyle\sum_{k=2}^{N-1} \sum_{l=k+1}^{N} \sum_{i=2}^{N-1} \sum_{j=i+1}^{N} d_{kl} f_{ij} x_{ki} x_{lj}$

$= \displaystyle\sum_{k=2}^{N-1} \sum_{l=k+1}^{N} d_{kl} \sum_{i=2}^{N-1} \left\{ \sum_{j=i+1}^{N-1} f_{ij} x_{ki} x_{lj} + f_{iN} x_{ki} x_{lN} \right\}$

$= \displaystyle\sum_{k=2}^{N-1} \sum_{l=k+1}^{N} d_{kl} \sum_{i=2}^{N-1} \left\{ \sum_{j=i+1}^{N-1} f_{ij} x_{ki} x_{lj} + f_{iN} x_{ki} \left(1 - \sum_{j=2}^{N-1} x_{lj} \right) \right\}$ (using eqn (5))

$= \displaystyle\sum_{k=2}^{N-1} \sum_{l=k+1}^{N} d_{kl} \sum_{i=2}^{N-1} f_{iN} x_{ki} + \sum_{k=2}^{N-1} \sum_{l=k+1}^{N} d_{kl} \sum_{i=2}^{N-1} \sum_{j=i+1}^{N-1} (f_{ij} - f_{iN}) x_{ki} x_{lj}$

$= \displaystyle\sum_{k=2}^{N-1} \sum_{i=2}^{N-1} \sum_{l=k+1}^{N} d_{kl} f_{iN} x_{ki} + \sum_{k=2}^{N-1} \sum_{i=2}^{N-1} \sum_{j=i+1}^{N-1} \left\{ \sum_{l=k+1}^{N-1} d_{kl} (f_{ij} - f_{iN}) x_{ki} x_{lj} + \right.$

$\qquad\qquad\qquad \left. d_{kN} (f_{ij} - f_{iN}) x_{ki} x_{Nj} \right\}$

$$= \sum_{k=2}^{N-1} \sum_{i=2}^{N-1} \sum_{l=k+1}^{N} d_{kl} f_{iN} x_{ki} + \sum_{k=2}^{N-1} \sum_{i=2}^{N-1} \sum_{j=i+1}^{N-1} \left\{ \sum_{l=k+1}^{N-1} d_{kl} (f_{ij} - f_{iN}) x_{ki} x_{lj} + \right.$$

$$\left. d_{kN} (f_{ij} - f_{iN}) x_{ki} \left(1 - \sum_{l=2}^{k} x_{lj} - \sum_{l=k+1}^{N-1} x_{lj} \right) \right\} \quad \text{(using eqn (7))}$$

$$= \sum_{k=2}^{N-1} \sum_{i=2}^{N-1} \sum_{l=k+1}^{N} f_{iN} d_{kl} x_{ki} + \sum_{k=2}^{N-1} \sum_{i=2}^{N-1} \sum_{j=i+1}^{N-1} d_{kN} (f_{ij} - f_{iN}) x_{ki} +$$

$$\sum_{k=2}^{N-1} \sum_{i=2}^{N-1} \sum_{j=i+1}^{N-1} \sum_{l=k+1}^{N-1} (d_{kl} - d_{kN})(f_{ij} - f_{iN}) x_{ki} x_{lj} -$$

$$\sum_{k=2}^{N-1} \sum_{i=2}^{N-1} \sum_{j=i+1}^{N-1} \sum_{l=2}^{k} d_{kN} (f_{ij} - f_{iN}) x_{ki} x_{lj}. \tag{11}$$

Using expressions (10) and (11) for Z_1 and Z_r respectively, the cost function Z for the circular ordering problem can be written as

$$Z = Z_1 + 2Z_r$$

$$= \sum_{l=2}^{N} d_{1l} f_{1N} + \sum_{j=2}^{N-1} d_{1N} (f_{1j} - f_{1N}) +$$

$$\sum_{j=2}^{N-1} \sum_{l=2}^{N-1} (d_{1l} - d_{1N})(f_{1j} - f_{1N}) x_{lj} + 2 \sum_{k=2}^{N-1} \sum_{i=2}^{N-1} \sum_{l=k+1}^{N} f_{iN} d_{kl} x_{ki} + 2 \sum_{k=2}^{N-1} \sum_{i=2}^{N-1} \sum_{j=i+1}^{N-1} d_{kN} (f_{ij} - f_{iN}) x_{ki} +$$

$$2 \sum_{k=2}^{N-1} \sum_{i=2}^{N-1} \sum_{j=i+1}^{N-1} \sum_{l=k+1}^{N-1} (d_{kl} - d_{kN})(f_{ij} - f_{iN}) x_{ki} x_{lj} - 2 \sum_{k=2}^{N-1} \sum_{i=2}^{N-1} \sum_{j=i+1}^{N-1} \sum_{l=2}^{k} d_{kN} (f_{ij} - f_{iN}) x_{ki} x_{lj}.$$

Since the terms $\sum_{l=2}^{N} d_{1l} f_{1N}$ and $d_{1N} \sum_{j=2}^{N-1} (f_{1j} - f_{1N})$ are independent of x_{ki} values, the circular ordering problem is the problem of minimizing the following non-linear function subject to the constraint sets (6) and (8).

$$Z = \sum_{j=2}^{N-1} \sum_{l=2}^{N-1} (d_{1l} - d_{1N})(f_{1j} - f_{1N}) x_{lj} + 2 \sum_{k=2}^{N-1} \sum_{i=2}^{N-1} \sum_{l=k+1}^{N} f_{iN} d_{kl} x_{ki} + 2 \sum_{k=2}^{N-1} \sum_{i=2}^{N-1} \sum_{j=i+1}^{N-1} d_{kN} (f_{ij} - f_{iN}) x_{ki} +$$

$$2 \sum_{k=2}^{N-1} \sum_{i=2}^{N-1} \sum_{j=i+1}^{N-1} \sum_{l=k+1}^{N-1} (d_{kl} - d_{kN})(f_{ij} - f_{iN}) x_{ki} x_{lj} - 2 \sum_{k=2}^{N-1} \sum_{i=2}^{N-1} \sum_{j=i+1}^{N-1} \sum_{l=2}^{k} d_{kN} (f_{ij} - f_{iN}) x_{ki} x_{lj}. \tag{12}$$

3 An Example

Consider an instance of circular ordering in which we have to order five facilities around a circle. The frequency of communication between pairs of facilities is given in the table below.

$F = [f_{ij}]$	1	2	3	4	5
1	-	5	2	7	8
2	5	-	6	4	3
3	2	6	-	9	1
4	7	4	9	-	7
5	8	3	1	7	-

Without loss of generality, we assume that the radius of the circle is $5/2\pi$ units so that the distance between consecutive positions is 1 unit. We also assume that facility 1 is located at position 1 (as specified in MODEL2).

In this example, the term Z_1 when evaluated without making use of equations (5) and (7) is given by

$$
\begin{aligned}
Z_1 = \ & 5x_{22} + 2x_{23} + \ \ 7x_{24} + ` \ 8x_{25} + \\
& 10x_{32} + 4x_{33} + 14x_{34} + 16x_{35} + \\
& 10x_{42} + 4x_{43} + 14x_{44} + 16x_{45} + \\
& 5x_{52} + 2x_{53} + \ \ 7x_{54} + \ \ 8x_{55}.
\end{aligned}
$$

From (5) $x_{25} = 1 - (x_{22} + x_{23} + x_{24})$; $x_{35} = 1 - (x_{32} + x_{33} + x_{34})$; $x_{45} = 1 - (x_{42} + x_{43} + x_{44})$; and $x_{55} = 1 - (x_{52} + x_{53} + x_{54})$. Substituting these terms,

$$Z_1 = 48 - 3x_{22} - 6x_{23} - x_{24} - 6x_{32} - 12x_{33} - 2x_{34} - 6x_{42} - 12x_{43} - 2x_{44} - 3x_{52} - 6x_{53} - x_{54}.$$

Next from (7) $x_{52} = 1 - (x_{22} + x_{32} + x_{42})$; $x_{53} = 1 - (x_{23} + x_{33} + x_{43})$; and $x_{54} = 1 - (x_{24} + x_{34} + x_{44})$. Substituting these terms,

$$Z_1 = 38 - 3x_{32} - 6x_{33} - x_{34} - 3x_{42} - 6x_{43} - x_{44}; \tag{13}$$

corresponding to expression (10).

The term Z_r when evaluated without making use of equations (5) and (7) is given by

$$
\begin{aligned}
Z_r = \ & 6x_{22}x_{33} + 4x_{22}x_{34} + 3x_{22}x_{35} + 9x_{23}x_{34} + 1x_{23}x_{35} + 7x_{24}x_{35} + \\
& 12x_{22}x_{43} + 8x_{22}x_{44} + 6x_{22}x_{45} + 18x_{23}x_{44} + 2x_{23}x_{45} + 14x_{24}x_{45} + \\
& 12x_{22}x_{53} + 8x_{22}x_{54} + 6x_{22}x_{55} + 18x_{23}x_{54} + 2x_{23}x_{55} + 14x_{24}x_{55} + \\
& 6x_{32}x_{43} + 4x_{32}x_{44} + 3x_{32}x_{45} + 9x_{33}x_{44} + 1x_{33}x_{45} + 7x_{34}x_{45} + \\
& 12x_{32}x_{53} + 8x_{32}x_{54} + 6x_{32}x_{55} + 18x_{33}x_{54} + 2x_{33}x_{55} + 14x_{34}x_{55} + \\
& 6x_{42}x_{53} + 4x_{42}x_{54} + 3x_{42}x_{55} + 9x_{43}x_{54} + 1x_{43}x_{55} + 7x_{44}x_{55}.
\end{aligned}
$$

Using equations (5) to eliminate x_{35}, x_{45}, and x_{55}, we have

$$
\begin{aligned}
Z_r = \ & 15x_{22} + 5x_{23} + 35x_{24} + 9x_{32} + 3x_{33} + 21x_{34} + 3x_{42} + x_{43} + 7x_{44} + \\
& 3x_{22}x_{33} + x_{22}x_{34} + 8x_{23}x_{34} + 6x_{22}x_{43} + 2x_{22}x_{44} + 16x_{23}x_{44} + 6x_{22}x_{53} + 2x_{22}x_{54} + \\
& 16x_{23}x_{54} + 3x_{32}x_{43} + x_{32}x_{44} + 8x_{33}x_{44} + 6x_{32}x_{53} + 2x_{32}x_{54} + 16x_{33}x_{54} + 3x_{42}x_{53} + \\
& x_{42}x_{54} + 8x_{43}x_{54} - 3x_{22}x_{32} - 7x_{24}x_{32} - 7x_{24}x_{33} - 7x_{24}x_{34} - 6x_{22}x_{42} - 2x_{23}x_{42} - \\
& 2x_{23}x_{43} - 14x_{24}x_{42} - 14x_{24}x_{43} - 14x_{24}x_{44} - 6x_{22}x_{52} - 2x_{23}x_{52} - 2x_{23}x_{53} - 14x_{24}x_{52} - \\
& 14x_{24}x_{53} - 14x_{24}x_{54} - 3x_{32}x_{42} - 7x_{34}x_{42} - 7x_{34}x_{43} - 7x_{34}x_{44} - 6x_{32}x_{52} - 2x_{33}x_{52} - \\
& 2x_{33}x_{53} - 14x_{34}x_{52} - 14x_{34}x_{53} - 14x_{34}x_{54} - 3x_{42}x_{52} - 7x_{44}x_{52} - 7x_{44}x_{53} - 7x_{44}x_{54} - \\
& x_{23}x_{32} - x_{23}x_{33} - x_{33}x_{42} - x_{33}x_{43} - x_{43}x_{52} - x_{43}x_{53}.
\end{aligned}
$$

Next using equations (7) to eliminate x_{52}, x_{53}, and x_{54}, and noting that for any binary variable y, $y^2 = y$, we get

$$
\begin{aligned}
Z_r = \ & 23x_{22} + 19x_{23} + 7x_{24} + 17x_{32} + 17x_{33} - 7x_{34} + 7x_{42} + 8x_{43} - 7x_{44} - \\
& x_{22}x_{33} + 13x_{22}x_{34} + 6x_{23}x_{34} + x_{22}x_{43} + 7x_{22}x_{44} + 7x_{23}x_{44} - 2x_{32}x_{43} + 6x_{32}x_{44} - x_{33}x_{44} + \\
& 9x_{22}x_{32} + 5x_{24}x_{32} - 9x_{24}x_{33} + 21x_{24}x_{34} + 3x_{22}x_{42} - 3x_{23}x_{42} + x_{23}x_{43} - x_{24}x_{42} - 8x_{24}x_{43} + \\
& 7x_{24}x_{44} + 6x_{32}x_{42} + 6x_{34}x_{42} - x_{34}x_{43} + 14x_{34}x_{44} - 5x_{23}x_{32} + 3x_{23}x_{33} - 2x_{33}x_{42} + 2x_{33}x_{43} + \\
& 6x_{44}x_{42} - x_{44}x_{43} - 2x_{43}x_{42} - 4x_{22}x_{23} + 12x_{22}x_{24} - 2x_{23}x_{24} - 4x_{33}x_{32} + 12x_{34}x_{32} - 2x_{34}x_{33}.
\end{aligned}
$$

This corresponds to expression (11).

So the cost function for the instance is given by

$$
\begin{aligned}
Z =&Z_1 + 2Z_r \\
=&38 + 46x_{22} + 38x_{23} + 14x_{24} + 31x_{32} + 28x_{33} - 15x_{34} + 11x_{42} + 10x_{43} - 15x_{44} - \\
=&2x_{22}x_{33} + 26x_{22}x_{34} + 12x_{23}x_{34} + 2x_{22}x_{43} + 14x_{22}x_{44} + 14x_{23}x_{44} - 4x_{32}x_{43} + 12x_{32}x_{44} - 2x_{33}x_{44} + \\
=&18x_{22}x_{32} + 10x_{24}x_{32} - 18x_{24}x_{33} + 42x_{24}x_{34} + 6x_{22}x_{42} - 6x_{23}x_{42} + 2x_{23}x_{43} - 2x_{24}x_{42} - 16x_{24}x_{43} + \\
=&14x_{24}x_{44} + 12x_{32}x_{42} + 12x_{34}x_{42} - 2x_{34}x_{43} + 28x_{34}x_{44} - 10x_{23}x_{32} + 6x_{23}x_{33} - 4x_{33}x_{42} + 4x_{33}x_{43} + \\
=&12x_{44}x_{42} - 2x_{44}x_{43} - 4x_{43}x_{42} - 8x_{22}x_{23} + 24x_{22}x_{24} - 4x_{23}x_{24} - 8x_{33}x_{32} + 24x_{34}x_{32} - 4x_{34}x_{33};
\end{aligned}
$$

and the solution to the instance will be the minimizer of this function subject to the constraints

$$
\sum_{i=2}^{4} x_{ki} \leq 1 \text{ for each } k = 2, \ldots, 4;
$$

$$
\sum_{k=2}^{4} x_{ki} \leq 1 \text{ for each } i = 2, \ldots, 4;
$$

$$
\sum_{k=2}^{4}\sum_{i=2}^{4} x_{k1} \geq 2;
$$

and the condition that all x_{ij} variables are binary variables.

4 Summary

In this paper we formulated a bi-directional loop layout problem with equidistant facilities as a quadratic programming problem. Using the quadratic assignment problem as a base, we use the symmetry properties of the problem to simplify the cost function. We also convert the constraints of the problem from the more restrictive equality constraints to the less restrictive inequality constraints. We provide an example of our formulation on a problem with five facilities and five candidate locations.

The formulation presented here is amenable to the use of preprocessing rules and solution through branch and bound procedures. These will be interesting directions for further research on this problem.

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