Multiple products, multiple constraints, single period inventory problem: A Hierarchical solution procedure

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Abstract

This paper presents the formulation and a hierarchical solution procedure of multiple products, multiple constraints, single period inventory problem. The hierarchical procedure decomposes the problem into a number of sub-problems equal to the number of constraints sets. Each sub-problem is solved optimally by applying Lagrange multipliers and satisfying Kuhn-Tucker conditions. The experimental results show that the hierarchical procedure performs well even when there are large a number of products and constraints.

Keywords: Newsboy problem, Single-period inventory problem

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1. Introduction

Single period inventory model, also popularly known as the *Newsboy* or *Christmas tree* problem, is to obtain a products' decision quantity that maximizes / minimizes the expected profit / loss under stochastic demand. One of the earliest works in this area was that by Hadley and Whittin (1963), and Hodges and Moore (1970). The later authors solved the product-mix problem with stochastic demand competing for a number of limited resources. Many of the earlier research papers dealt with the objective of maximizing the probability of achieving a target profit [Kabak and Schiff (1978), Shih (1979), Lau (1980), etc.].

There has been much extension to the classical single period inventory problems registered after 80's. One major extension area is to use optimization techniques to solve multi-product, multi-constraint single period inventory problem. Khouja (1999) in his review paper has classified the extension in eleven categories. One more extension area in addition to the eleven categories may be applying different techniques to solve the problem; like marginal analysis [Hodges and Moore (1970)], Lagrange multiplier [Karmarkar (1981), and Lau and Lau (1995), Lau and Lau (1997), etc.], heuristic method [Nahmias and Schmidt (1984), etc.], analytical solution procedure [Ben-Daya and Raouf (1993), etc.].

Much of the interest in single period inventory problem started with the introduction of multi-period, multi-constraint problem. Several authors including [Hadley and Whittin (1963), Nahmias and Schmidt (1984), Lau and Lau (1995), Lau and Lau (1996), and Vairaktarakis (2000), etc.] solved the single-period multi-product constrained inventory models. The constraint sets often considered by these authors are mainly storage space, production capacity, and budget. The methodologies used to solve these problems are quite different. Hadley and Whittin (1963) have solved a single constraint set problem by Lagrangian multiplier and the solution procedure was suitable for large quantities. They have adopted marginal analysis approach to solve for small quantities. Nahmias and Schmidt (1984) pointed out that Lagrangian method may require higher computation. They have developed four different heuristics to solve single constraint problem. Lau and Lau (1996) have solved multiple constraint problems. They converted N-variable 'primal problem' to 'M' variable 'dual problem' and developed 'active set methods' to solve the problem. The solution procedure developed by Lau and Lau (1996) will perform well only with low number of dual variables i.e. constraints. But, there are many situations where the number of constraints is large and active set method may fail to provide an efficient solution.

In this research paper, we have attempted to develop a hierarchical method of solving single-period multiple product inventory problem with a large number of constraints. Section 2 and 3 describes the problem formulation and solution methodology respectively. Section 4 deals with the hierarchical solution procedure with a detailed example of convergence. Section 5 provides computational experiments for convergence. Section 6 deals with the extension of the solution procedure to three constraint sets and section 7 summarizes our paper.

2. Problem formulation

We came across a practical multi-product, multi-constraint problem while dealing with a dairy in a large city. The dairy offers four different kind of milk and sells it through more than 100 retail outlets in the city. All these retail outlets face stochastic demand of different types of milk. These outlets meet the demand from their stock, which is replenished once in a day. In addition, the outlets have storage space constraint. The retail outlets generally face stock-out and excess inventory situation. Both stock-out and inventory have associated understocking and overstocking cost per unit shortage and excess inventory. In addition, there is limited available supply of each type of milk at the dairy.

2.1 Notation

The notation used in the model formulation is as follows:

- r: Set of retailers (1 to R)
- p: Set of products (1 to P)
- x: A random variable representing the demand
- $f_{rp}(x)$: Probability density-function of demand of product 'p' at retail outlet 'r'
- US_p: Understocking cost of product 'p' (Rs. per unit)
- OS_p: Overstocking cost of product 'p' (Rs. per unit)
- Cap_r: Storage capacity at retail outlet 'r' (units)
- Sup_p: Available supply of product 'p' (units)
- E_{rp}: Expected cost of product 'p' at retail outlet 'r' when supply quantity is Q_{rp}.
- μ_{rp}: Mean demand of product 'p' at retailer 'r'.
- σ_{rp}: Standard deviation of product 'p' at retailer 'r'.
- $z_{rp}()$: Standard normal deviation of product 'p' at retailer 'r'.
- $\Phi_{rp}()$: Cumulative density function of product 'p' at retailer 'r'.
- $\varphi_{rp}()$: Probability density function of product 'p' at retailer 'r'.

The planning horizon is a day. Therefore, the parameter units viz. demand, supply is for a day. Also, we have taken the demand as normal distributed.

2.2 Objective function

The objective is to minimize the total expected understocking and overstocking cost for all products at all retail outlets.

Z_1: Minimize $_{(Q_{rp})}$

$$\sum_p \Big(\sum_r \Big(\int_0^{Q_{rp}} (Q_{rp} - x).f_{rp}(x).dx\Big) * OS_p\Big) + \sum_p \Big(\sum_r \Big(\int_{Q_{rp}}^{\infty} (x - Q_{rp}).f_{rp}(x).dx\Big) * US_p\Big) \qquad ..1$$

2.3 Constraint sets

$$\sum_p Q_{rp} \le Cap_r \qquad r \in \{1\ldots R\} \qquad ..2$$

$$\sum_r Q_{rp} \le Sup_p \qquad p \in \{1\ldots P\} \qquad ..3$$

$$Q_{rp} \ge 0 \qquad r \in \{1\ldots R\},\ p \in \{1\ldots P\} \qquad ..4$$

Constraint 2 deals with the storage capacity constraint of the retail outlets. The storage space constraint is the physical / refrigerator space constraint at the retail outlets. Constraint 3 ensures that the delivery does not exceed available supply of the product, and constraint 4 is a non-negativity constraint.

3. Solution methodology

The above-mentioned problem has a convex objective function [Federgruen and Zipkin (1984)] with linear constraint. Thus, both the objective function and the constraint sets are differentiable functions. The objective function is differentiable as equating the partial differential of equation 1 with respect to Q_{rp} to zero gives us $Q_{rp} = F_{rp}^{-1}[US_p / (US_p + OS_p)]$. We can solve this problem optimally by relaxing the constraints by Lagrange multipliers and then applying the Kuhn-Tucker conditions [Shapiro (1979)]. We take λ_r, δ_p and η_{rp} as Lagrange multipliers for the constraints 2, 3 and 4 respectively. The relaxed objective function is given by equation 5.

L (λ, δ, η) = Minimize $_{(Qrp)}$

$$\sum_p (\sum_r (\int_0^{Q_{rp}} (Q_{rp} - x).f_{rp}(x).dx) * OS_p) + \sum_p (\sum_r (\int_{Q_{rp}}^{\infty} (x - Q_{rp}).f_{rp}(x).dx) * US_p)$$
$$+ \sum_r \lambda_r * (\sum_p Q_{rp} - Cap_r) + \sum_p \delta_p * (\sum_r Q_{rp} - Sup_p) - \sum_r \sum_p (\eta_{rp} * Q_{rp}) \quad ..5$$

Where: $\lambda_r \geq 0$, $\delta_p \geq 0$ and $\eta_{rp} \geq 0$.

From global optimality conditions [Shapiro (1979)], we have for any $(\lambda_r, \delta_p, \eta_{rp}) \geq 0$, L (λ, δ, η) ≤ Z_1. Thus, all the objective function and the variables are at their optimal values when L (λ, δ, η) = Z_1. The corresponding optimal values of dual variables λ_r, δ_p, and η_{rp} can be found by using Kuhn-Tucker conditions.

3.1 Simultaneous equations by Kuhn-Tucker conditions

The sets of simultaneous equations derived from Kuhn-Tucker conditions to solve the primal problem are as follows:

$$Q_{rp} = F_{rp}^{-1}\left[(US_p - \lambda_r - \delta_p + \eta_{rp})/(US_p + OS_p)\right] \quad r \in \{1\ldots R\},\ p \in \{1\ldots P\} \quad ..6$$

$$\lambda_r * \left[\sum_p F_{rp}^{-1}\left[(US_p - \lambda_r - \delta_p + \eta_{rp})/(US_p + OS_p)\right] - Cap_r\right] = 0 \quad r \in \{1\ldots R\} \quad ..7$$

$$\sum_p F_{rp}^{-1}\left[(US_p - \lambda_r - \delta_p + \eta_{rp})/(US_p + OS_p)\right] - Cap_r \leq 0 \quad r \in \{1\ldots R\} \quad ..8$$

$$\delta_p * \left[\sum_r F_{rp}^{-1}\left[(US_p - \lambda_r - \delta_p + \eta_{rp})/(US_p + OS_p)\right] - Sup_p\right] = 0 \quad p \in \{1\ldots P\} \quad ..9$$

$$\sum_r F_{rp}^{-1}\left[(US_p - \lambda_r - \delta_p + \eta_{rp})/(US_p + OS_p)\right] - Sup_p \leq 0 \quad p \in \{1\ldots P\} \quad ..10$$

$$\eta_{rp} * \left[F_{rp}^{-1}\left[(US_p - \lambda_r - \delta_p + \eta_{rp})/(US_p + OS_p)\right]\right] = 0 \quad r \in \{1\ldots R\},\ p \in \{1\ldots P\} \quad ..11$$

$$F_{rp}^{-1}\left[\left(US_p - \lambda_r - \delta_p + \eta_{rp}\right)/\left(US_p + OS_p\right)\right] \geq 0 \qquad r \in \{1 \ldots R\}, p \in \{1 \ldots P\} \qquad ..12$$

$$\lambda_r \geq 0 \qquad r \in \{1 \ldots R\} \qquad ..13$$

$$\delta_p \geq 0 \qquad p \in \{1 \ldots P\} \qquad ..14$$

$$\eta_{rp} \geq 0 \qquad r \in \{1 \ldots R\}, p \in \{1 \ldots P\} \qquad ..15$$

Simplifying the above sets of simultaneous equations and inequalities of Kuhn-Tucker conditions, we have to only solve for minimum positive values of λ_r, δ_p, and η_{rp}, which are given by equations 17, 18 and 19. Whenever there is slack in the constraint, the corresponding dual value is equal to zero. Thus, whenever the minimum positive values of λ_r, δ_p and η_{rp} satisfies equations 17, 18 and 19 respectively, the values of λ_r, δ_p and η_{rp} are optimal. These optimal values of λ_r, δ_p and η_{rp} provides optimal Q_{rp} by putting the values in equation 16, which is obtained by partially differentiating equation 5 with respect to Q_{rp} and is a function of λ_r, δ_p and η_{rp}. The role of η_{rp} is to restrict the value of $(US_p - \lambda_r - \delta_p + \eta_{rp})$ to be non-negative.

$$Q_{rp} = F_{rp}^{-1}\left[\left(US_p - \lambda_r - \delta_p + \eta_{rp}\right)/\left(US_p + OS_p\right)\right] \qquad r \in \{1 \ldots R\}, p \in \{1 \ldots P\} \qquad ..16$$

$$\sum_p F_{rp}^{-1}\left[\left(US_p - \lambda_r - \delta_p + \eta_{rp}\right)/\left(US_p + OS_p\right)\right] - Cap_r = 0 \qquad r \in \{1 \ldots R\} \qquad ..17$$

$$\sum_r F_{rp}^{-1}\left[\left(US_p - \lambda_r - \delta_p + \eta_{rp}\right)/\left(US_p + OS_p\right)\right] - Sup_p = 0 \qquad p \in \{1 \ldots P\} \qquad ..18$$

$$F_{rp}^{-1}\left[\left(US_p - \lambda_r - \delta_p + \eta_{rp}\right)/\left(US_p + OS_p\right)\right] = 0 \qquad r \in \{1 \ldots R\}, p \in \{1 \ldots P\} \qquad ..19$$

The above multiple constrained problems can be optimally solved by the sub-gradient optimization technique [Shapiro (1979)]. However, using this technique to find the values of (R+P) vectors is computationally difficult. Therefore, we have adopted a hierarchical method to solve the problem.

4. Hierarchical solution procedure

In this method, we have broken the problem into two sub-problems. Each sub-problem deals with a single set of constraints. The single constraint problem can be optimally solved by binary search method. The first sub-problem (20 to 22) is to solve for λ_r for a given value of δ_p. The expression for Q_{rp} is given by equation 20. For a given values of δ_p, the minimum values of λ_r satisfying equation 21 can be found by the binary search method. Whenever the sum of the values of λ_r and δ_p exceeds US_p, η_{rp} takes the minimum value to make the numerator non-negative and to satisfy equation 22. Thus, the corresponding quantity Q_{rp} is considered to be equal to zero. Else, we can obtain the value of Q_{rp} by putting the values of λ_r and δ_p in equation 20.

$$Q_{rp} = F_{rp}^{-1}\left[\left(US_p - \lambda_r - \delta_p + \eta_{rp}\right)/\left(US_p + OS_p\right)\right] \qquad r \in \{1 \ldots R\},\ p \in \{1 \ldots P\} \qquad ..20$$

$$\sum_p F_{rp}^{-1}\left[\left(US_p - \lambda_r - \delta_p + \eta_{rp}\right)/\left(US_p + OS_p\right)\right] - Cap_r = 0 \qquad r \in \{1 \ldots R\} \qquad ..21$$

$$F_{rp}^{-1}\left[\left(US_p - \lambda_r - \delta_p + \eta_{rp}\right)/\left(US_p + OS_p\right)\right] = 0 \qquad r \in \{1 \ldots R\},\ p \in \{1 \ldots P\} \qquad ..22$$

The second sub-problem (23 to 25) is to solve for δ_p for given value of λ_r. The expression for replenishment quantities is given by 23. For a given values of λ_r, the minimum values of δ_p satisfying equation 24 can be found by binary search method. As discussed in sub-problem 1, whenever the sum of the values of λ_r and δ_p exceeds US_p, η_{rp} takes the value to make the numerator non-negative and satisfy equation 25. Thus, the corresponding quantity Q_{rp} is considered to be equal to zero. Else, we can obtain the value of Q_{rp} by putting the values of λ_r and δ_p in equation 23.

$$Q_{rp} = F_{rp}^{-1}\left[\left(US_p - \lambda_r - \delta_p + \eta_{rp}\right)/\left(US_p + OS_p\right)\right] \qquad r \in \{1 \ldots R\},\ p \in \{1 \ldots P\} \qquad ..23$$

$$\sum_r F_{rp}^{-1}\left[\left(US_p - \lambda_r - \delta_p + \eta_{rp}\right)/\left(US_p + OS_p\right)\right] - Sup_p = 0 \qquad p \in \{1 \ldots P\} \qquad ..24$$

$$F_{rp}^{-1}\left[\left(US_p - \lambda_r - \delta_p + \eta_{rp}\right)/\left(US_p + OS_p\right)\right] = 0 \qquad r \in \{1 \ldots R\},\ p \in \{1 \ldots P\} \qquad ..25$$

The two sub-problems are solved one after the other iteratively and updating of the values of λ_r and δ_p. The procedure of the iterative approach is discussed in the following sub-section.

4.1 Iterative procedure for solving the problem

We will solve for λ_r keeping δ_p value zero using equations 20, 21, and 22. We will again solve for δ_p keeping λ_r value zero using equations 23, 24, and 25. We will calculate the quantities and the respective expected cost and then compare the total expected cost for the two sub-problems. The expected cost associated with any product and any retail outlet can be calculated by equation 26 [refer Lau (1997) for derivation]. The expected cost function can be given by:

$$E_{rp}(Q_{rp}) = OS_p * \left(\int_0^{Q_{rp}} \left(Q_{rp} - x\right) * f_{rp}(x) * dx \right) + US_p * \left(\int_{Q_{rp}}^{\infty} \left(x - Q_{rp}\right) * f_{rp}(x) * dx \right)$$

Solving, we have

$$E_{rp}(Q_{rp}) = \sigma_{rp} * \left[- US_p * z_{rp} + (US_p + OS_p) * \left\{ z_{rp} * \phi_{rp}(z_{rp}) + \varphi_{rp}(z_{rp}) \right\} \right] \qquad ..26$$

The sub-problem having higher value of total expected cost will be called first so as to start with the tighter lower bound. Suppose the total expected cost with δ_p is high, then we will follow:

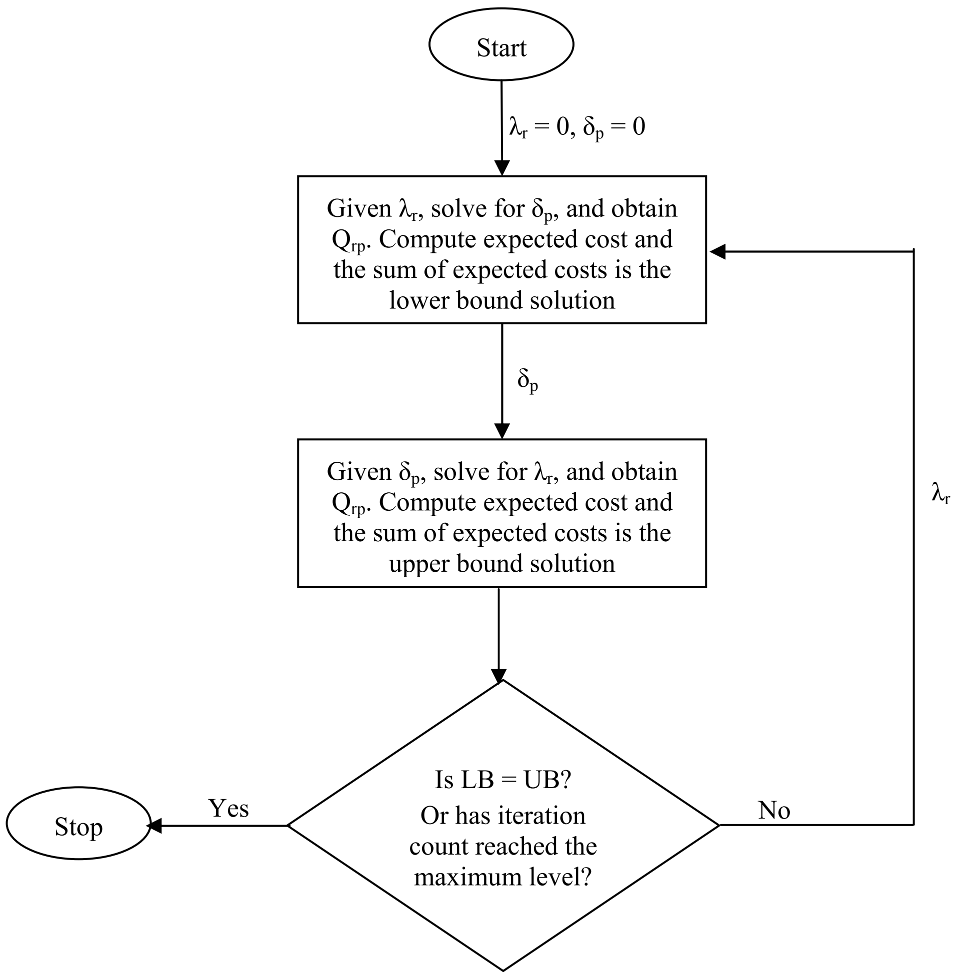

Figure 1: Hierarchical method for solving two constraints set problem

In this method, the first sub-problem will provide the lower bound solution and the second sub-problem will provide the upper bound solution. Since, it is a convex problem and has a unique solution, the lower bound will monotonically increase and upper bound will monotonically decrease and finally both will converge to the optimal solution. After each iteration, the dual variable values (either δ_p or λ_r) of first sub-problem will decrease and dual variable values of second sub-problem will increase. The dual variables and the quantities will attain the optimal value when the lower bound and upper bound solution converges.

4.2 Explanation

Lemma:

After each iteration, the Lagrange multiplier value / dual variable values (λ_r or δ_p) of first sub-problem will decrease and that of second sub-problem will increase.

Proof:

Suppose the first sub-problem deals with capacity constraint (λ_r) and second sub-problem deals with supply constraint (δ_p). In the first iteration, $\delta_p = 0$ and λ_r will take minimum non-negative value to satisfy capacity constraint. Based on the obtained λ_r value, δ_p will take a non-negative value to satisfy the supply constraint. Now, in the second iteration δ_p will have non-negative value and thus a lower value of λ_r may satisfy the capacity constraints. The lower value of λ_r will lead to a higher value of δ_p to satisfy the supply constraint. Thus, after each iteration the values of λ_r will decrease and the values of δ_p will increase.

It is evident from the above explanation that the dual variable values monotonically leads towards the optimal values. Also, we know that the convex problems have a unique optimal solution. From these two statements we can state that the lower bound solution will monotonically increase and the upper bound solution will monotonically decrease and will finally converge.

4.3 Example

We have taken an example of two retail outlet and two products. The parameters are provided in the following table.

Table 1: Parameter values for example problem

	R1-P1	R1-P2	R2-P1	R2-P2
Mean demand	20	25	25	20
Standard deviation	2	4	3	5
Overstocking cost	1	2	1	2
Understocking cost	4	5	4	5

The storage capacities at retailer-1 and retailer-2 are 40 and 45 respectively. The supplies of product-1 and product-2 are 40 and 45 respectively. We will consider Q_{rp} as the optimal quantity of product 'p' at retailer 'r', λ_r as the dual variable value of retailer 'r', and δ_p as the dual variable value of supply 'p'.

The total expected cost with capacity constraint is more than the total expected cost with supply constraint. Therefore, we will first call sub-problem solving capacity constraint and then call the sub-problem solving supply constraint.

4.4 Iterative steps

- For the given value of δ_p equal to zero, we solved for λ_r values and the values of λ_1 and λ_2 satisfying equation A4.11 are 3.3189 and 1.5000 respectively.
- Putting the values of dual values in equation A4.10, we obtained the quantity values. We have calculated the expected cost by putting the quantity value in equation 5.19. The sum of the expected cost is 24.63, which is the lower bound.

- For the given value of λ_1 and λ_2 equal to 3.3189 and 1.5000 respectively, we solved for δ_p values and the values of δ_1 and δ_2 satisfying equation A4.14 are 0.5860 and 0.0000 respectively.
- Putting the values of dual values in equation A4.13, we obtained the quantity values. The values of Q_{11}, Q_{12}, Q_{21}, and Q_{22} are 15.89, 22.19, 24.11 and 20.00 respectively. We have calculated the expected cost and the sum of the expected cost is 34.33, which is the upper bound. This completes Iteration 1.
- For the given value of δ_1 and δ_2 equal to 0.5860 and 0.0000 respectively, we solved for λ_r values and the values of λ_1 and λ_2 satisfying equation A4.11 are 2.9915 and 1.2321 respectively.
- Putting the values of dual values in equation A4.10, we obtained the quantity values. We have calculated the expected cost and the sum of the expected cost is 24.76, which is the lower bound.
- For the given value of λ_1 and λ_2 equal to 2.9915 and 1.2321 respectively, we solve for δ_p and the values of δ_1 and δ_2 satisfying equation A4.14 are 0.8993 and 0.0000 respectively.
- Putting the values of dual values in equation A4.13, we got the quantity values. The values of Q_{11}, Q_{12}, Q_{21}, and Q_{22} are 15.97, 22.75, 24.03 and 20.48 respectively. We have calculated the expected cost and the sum of the expected cost is 31.21, which is the upper bound. This completes Iteration 2.

The following table presents the quantities, Lagrange multiple, lower bound and upper bound values after each iteration. The values against iteration 1 shows after both the sub-problems are solved.

Table 2: Iterative values of convergence test

Iteration	Q_{11}	Q_{12}	Q_{21}	Q_{22}	λ_1	λ_2	δ_1	δ_2	LB	UB
1	15.89	22.19	24.11	20.00	3.3139	1.5000	0.5860	0.0000	24.63	34.33
2	15.97	22.75	24.03	20.48	2.9915	1.2321	0.8993	0.0000	24.76	31.21
3	16.04	23.08	23.96	20.74	2.7915	1.0880	1.0894	0.0000	25.19	29.60
4	16.10	23.29	23.90	20.90	2.6601	1.0003	1.2125	0.0000	25.58	28.66
5	16.14	23.43	23.86	21.00	2.5708	0.9430	1.2955	0.0000	25.89	28.06
6	16.17	23.52	23.83	21.07	2.5089	0.9042	1.3528	0.0000	26.13	27.68
7	16.19	23.59	23.81	21.12	2.4653	0.8775	1.3929	0.0000	26.31	27.41
8	16.20	23.64	23.80	21.16	2.4344	0.8587	1.4212	0.0000	26.44	27.23
9	16.22	23.67	23.78	21.18	2.4122	0.8454	1.4415	0.0000	26.53	27.10
10	16.22	23.69	23.78	21.20	2.3964	0.8359	1.4559	0.0000	26.60	27.01
11	16.23	23.71	23.77	21.21	2.3851	0.8291	1.4662	0.0000	26.65	26.95
12	16.23	23.72	23.77	21.22	2.3770	0.8243	1.4736	0.0000	26.69	26.90
13	16.24	23.73	23.76	21.23	2.3712	0.8208	1.4789	0.0000	26.71	26.87
14	16.24	23.74	23.76	21.23	2.3669	0.8183	1.4828	0.0000	26.73	26.84
15	16.24	23.74	23.76	21.24	2.3637	0.8165	1.4857	0.0000	26.74	26.83
16	16.24	23.75	23.76	21.24	2.3615	0.8151	1.4878	0.0000	26.75	26.81
17	16.24	23.75	23.76	21.24	2.3598	0.8141	1.4893	0.0000	26.76	26.81
18	16.24	23.75	23.76	21.24	2.3586	0.8134	1.4904	0.0000	26.77	26.80
19	16.24	23.75	23.76	21.24	2.3578	0.8129	1.4912	0.0000	26.77	26.79
20	16.24	23.75	23.76	21.24	2.3572	0.8126	1.4918	0.0000	26.78	26.78

The table shows that the lower bound is monotonically increasing and the upper bound is monotonically decreasing. After 20 iterations the lower bound and the upper bound converge and provide the optimal solution. The graph showing the convergence of lower bound and upper bound is as follows:

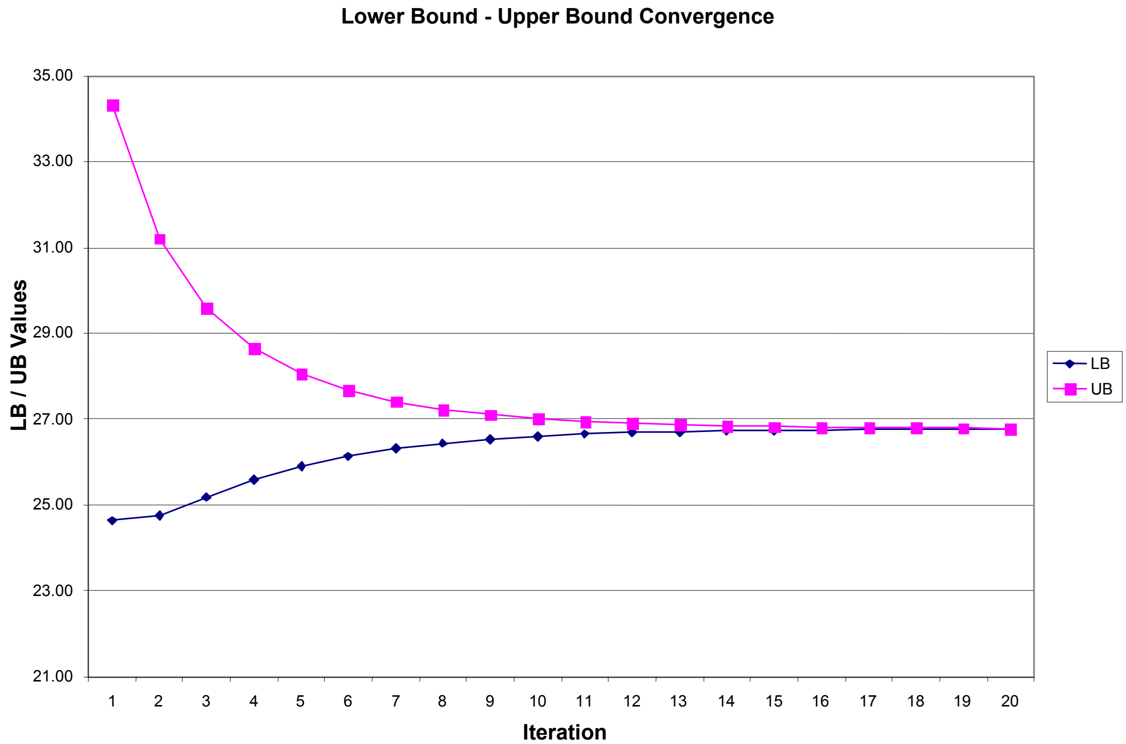

Figure 2: Graph showing iterative results and convergence

5. Experiments

We have implemented the hierarchical solution procedure in Visual Basic 6.0 with Excel interface. We have designed many experiments to test the quality of the solution. The objective of the experiments is to computationally verify the convergence of lower bound and the upper bound of the problem. The parameters to measure the performance of the hierarchical method are as follows:

- Percentage of instances converged.
- Average number of iterations required for convergence.
- Average percentage deviation between lower bound and upper bound.

- Average time required for convergence.

5.1 Experiment-1

In this experiment, we have considered

- 50 retailers and 4 products.
- Overstocking cost is considered 1 for all products. Understocking cost is considered as 2.0, 2.5, 2.5 and 3.0 for four different products.
- Mean demand is randomly generated between 10 and 40 for every (r, p) combination.
- 2 different scenarios of coefficient of variation, which is randomly generated between 0.1 and 0.25, and 0.1 and 0.4 for every (r, p) combination.
- 2 different scenarios of storage space capacity, which is randomly generated between 100 to 150 and 75 to 100 for every retailer.
- 2 different scenarios of supply, which is randomly generated between 1000 to 1500 and 500 to 800 for every product.

Thus, we have generated 8 test problems and for each test problem we have run 25 instances. We have also considered 100 as the maximum number of iteration. The results of these test problems are discussed in the following table:

Table 3: Convergence test results for Experiment-1

Test	Coeff. of variation	Storage capacity	Supply	Instances converged	Av no. of iterations	% deviation	Average time (seconds)
1	0.1 - 0.25	100 – 150	1000-1500	25	1.2	0.0%	0.20
2	0.1 - 0.25	100 – 150	500-800	25	5.8	0.0%	10.52
3	0.1 - 0.25	75 – 100	1000-1500	25	1.0	0.0%	2.44
4	0.1 - 0.25	75 – 100	500-800	24	13.5	0.5%	9.00
5	0.1 – 0.4	100 – 150	1000-1500	25	1.2	0.0%	0.12
6	0.1 – 0.4	100 – 150	500-800	25	18.8	0.0%	15.08
7	0.1 – 0.4	75 – 100	1000-1500	25	1.0	0.0%	1.28
8	0.1 – 0.4	75 – 100	500-800	23	46.7	0.76%	65.48

In this experiment, 197 out of 200 instances have converged. The maximum average deviation is 0.76%. The average number of iterations to converge is less than 20. The average time to solve the problem is less than 15 seconds. The average time in some tests is high as compared to the other tests because the constraints are very tight and thus the value of cumulative density function is even less than 0.1. For these low values of cumulative density function, we have calculated the respective normal value by Microsoft Excel (instead of using inverse CDF conversion table), which has taken more time.

5.2 Experiment-2

This experiment has been designed to capture identical retailers facing random demand with very low standard deviation. In this experiment, we have considered

- 50 retailers and 4 products
- Overstocking cost is considered 1 for all products. Understocking cost is randomly allocated (either of 3 values 1.0 / 1.5 / 2.0) for every instance.

- Mean demand is randomly generated between 10 to 30 for every (r, p) combination. Standard deviation of demand is randomly generated between 1 to 3 for every (r, p) combination.
- 4 different scenarios of storage space capacity are considered and are equal to 100 / 80 / 60 / 40 for all retailers. 4 different scenarios of supply are considered and are equal to 1000 / 800 / 600 / 400 for all products.

All other parameters are similar to the previous experiment. The results of these test problems are discussed in the following table:

Table 4: Convergence test results for Experiment-2

Test	Storage capacity	Supply	Instances converged	Average number of iteration	% deviation	Average time (seconds)
1	100	1000	25	1.00	0.0%	0.36
2	80	1000	25	1.00	0.0%	1.24
3	60	1000	25	1.00	0.0%	1.52
4	40	1000	25	1.00	0.0%	3.12
5	100	800	25	1.12	0.0%	0.76
6	80	800	25	1.92	0.0%	1.40
7	60	800	25	1.96	0.0%	4.68
8	40	800	25	1.40	0.0%	4.08
9	100	600	25	1.00	0.0%	3.08
10	80	600	25	1.00	0.0%	3.20
11	60	600	25	1.64	0.0%	4.24
12	40	600	25	1.00	0.0%	7.68
13	100	400	25	1.00	0.0%	6.36
14	80	400	25	1.00	0.0%	6.44
15	60	400	25	1.00	0.0%	6.76
16	40	400	23	6.72	3.7%	8.95

In this experiment, 398 out of 400 instances have been converged. The average number of iterations to converge is less than 2. The average time to solve the problem is less than 6 seconds. The average deviation of instances that did not converge is 3.7%.

5.3 Summary of the experimental results

The summary of the experimental results of the problem is as follows:

- The percentage of convergence is more than 99%. If it does not convergence, the maximum deviation from the lower bound is less than 4%.
- The number of iterations and time required for solving the problem increases with constraint tightness. All the non-converged instances were very tight in terms of constraint values.
- Increasing the number of iterations of non-convergence instances to very high values (typically 500-1000) may lead to convergence.

5.4 Worst-case Analysis

The performance of the hierarchical method deteriorates with the tightness of the constraints. This procedure will perform worst when the constraints value is less than 20% of the sum of unconstrained optimal quantity competing for the resource. The decrease in the constraint values leads to following:

- Increase in the number of iterations for convergence
- More time for convergence
- Higher probability that the problem will not converge
- Higher percentage deviation in lower bound and upper bound

However, most of the real-world constrained problems have resources at least 50% of the unconstrained optimal requirement. With resources above 50% of optimal requirement, our hierarchical method provides efficient solution [almost 100% convergence with average number of iterations less than 5 and average time required to solve being less than 5 seconds].

6. Three constraint set problem

In continuation of the earlier two constraints set problem, we will consider the case of three constraints set. In our situation let us consider that the retail outlets are divided into several zones and each zone is being serviced by a vehicle. The sum of the supplies to the retail outlets in each zone is constrained by the vehicle load capacity. We have taken ζ_v as the Lagrange multiplier for the vehicle load constraints and adopted the similar procedure as per figure 3.

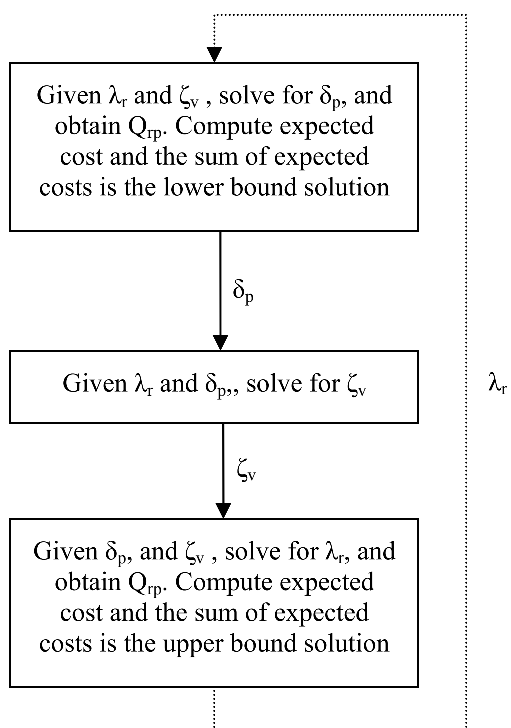

Figure 3: Hierarchical method for solving three constraints set problem

7. Conclusion

We have discussed the formulations and solution procedure for a multi-product multi-constraint single period inventory problem with two or more constraint sets. The proposed hierarchical solution procedure has provided efficient results for problems with large number of products and constraints. We have taken the product demand distribution as normal distribution, but the solution procedure can be used for any demand distribution.

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